The spectral flow in variational bifurcation theory

Nils Waterstraat

Abstract

A special feature in nonlinear analysis is the existence of multiple solutions, and bifurcation is a paradigm for this nonuniqueness. The basic idea of bifurcation theory is to study equations $F(\lambda, u) = 0$, where $F : \mathbb{R} \times X \to X$ is a sufficiently smooth family of maps on some Banach space X such that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. A *bifurcation point* is a parameter value λ^* at which non-trivial solutions branch off from the trivial ones $\mathbb{R} \times \{0\}$.

The first aim of this talk is to explain that the derivative $L_{\lambda^*} := D_0F(\lambda^*, \cdot) \in \mathcal{L}(X)$ is not invertible if λ^* is a bifurcation point, which follows from the implicit function theorem. Afterwards, we briefly review the classical Krasnoselski bifurcation theorem for maps of the type $F(\lambda, u) = \lambda u - G(u)$, where $D_0G : X \to X$ is assumed to be compact. Subsequently, we focus on Hilbert spaces and assume that $L_\lambda = \lambda\, id_H - D_0G$ is selfadjoint. The final result of this first part of the talk will express the well known principle that a jump in the Morse index of L_λ, when λ travels along the real line, implies the existence of a bifurcation point.

In the main part of this talk, we introduce the spectral flow for paths of selfadjoint Fredholm operators, which is an integer valued homotopy invariant that was introduced by Atiyah, Patodi and Singer in the seventies. We explain a bifurcation theorem of Fitzpatrick, Pejsachowicz and Recht from 1999 (in an improved version of Pejsachowicz (Politecnico di Torino) and myself from 2013), which treats more general equations $F(\lambda, u) = 0$ than the ones in Krasnoselski's theorem and shows that the non-vanishing of the spectral flow of the path $L_\lambda := D_0F(\lambda, \cdot)$, $\lambda \in \mathbb{R}$, entails the existence of a bifurcation point. We reobtain Krasnoselski's theorem in Hilbert spaces as a corollary of this result.

Finally, if time permits, we illustrate the bifurcation theorem by examples for elliptic semilinear PDE on star-shaped domains, where the bifurcation parameter is introduced by shrinking the domain to a point. These examples were developed in joint papers with A. Portaluri (Università degli studi di Torino) during the last two years.