SEMINARIO DI GEOMETRIA E ALGEBRA

UNIBA - POLIBA

Lunedì 29 Settembre 2025
Dipartimento di Matematica UniBa, Aula IX, primo piano

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e
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- **15:00–15:45, Swanhild Bernstein:**
  Q^2-Clifford algebras
- **16:00–16:45, Martha Lina Zimmerman:**
  Jackson Clifford Analysis and the q-Fock spaces

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Swanhild Bernstein
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Q^2 Clifford algebras

Abstract. Clifford analysis can be seen as a higher-dimensional function theory and a refinement of harmonic analysis. The building blocks are Clifford algebras and the Dirac operator. The Dirac operator is a first-order differential operator that factorizes the Laplacian. It is well-known that the Dirac operator is rotationally invariant, i.e., $SO(n)$-invariant.

The first "q" is introduced through the q-derivative or q-difference operator:

$$D_q f(x) := \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}.$$

In higher dimensions, we similarly define partial derivatives. This leads to different versions of q-Clifford analysis, one of which is close to analytic functions.

The second "q" stands for "quantum". Clifford algebras arise naturally in Quantum theory because they allow us to construct the Dirac operator, which factorizes the Laplacian in the quantum context, namely, a Hamiltonian. A real Clifford algebra is typically generated by n elements that satisfy the relation

$$e_i e_j + e_j e_i = -2\delta_{i,j}.$$

In the case of $2n$ real generating elements, it is possible to construct a so-called Witt basis, which fulfills a Grassmann relation, the duality relation, and isotropy conditions. This basis can be identified with fermionic raising and lowering operators that adhere to the canonical anticommutation relations.

In the quantum context, it is also natural to consider q-commutative variables instead of commuting variables; q-commuting variables fulfill

$$x_i x_j = -q x_j x_i, \quad i < j.$$

In this context, the quantum groups discovered by Drinfeld and Jimbo arise. The analog of the $SU(n)$ is the pseudogroup $SU_q(n)$ and the SU_q-invariant Laplacian is

$$\Delta_q = q^{n-1}(\partial_1^q)^2 + q^{n-2}(\partial_2^q)^2 + \ldots + (\partial_n^q)^2 = \sum_{i=1}^{n} q^{n-i}(\partial_i^q)^2.$$

Starting from these results, we will construct a Dirac operator that factorizes this Laplacian. As a result, we will need to consider new types of Clifford algebras that take the q-commutativity into account. We will prove a Cauchy-Kovalevskaja extension, a decomposition of q-harmonic functions and q-monogenic functions (kernel of the q-Dirac operator), i.e., a Fischer decomposition.

Another way to generalize Clifford analysis is going back to the Witt basis of a (complex) Clifford algebra stemming from the canonical anticommutation relations gives another alley to q-Clifford algebras. This is generalized to the twisted canonical anticommutation relations, which are described within a formalism for second quantization based on the pseudogroup $SU_q(n)$ where $0 < q < 1$. We will define a Dirac operator and demonstrate that this Dirac operator has different properties than the usually one.

Martha Zimmermann
TU Bergakademie Freiberg

Jackson Clifford Analysis and the q-Fock spaces

Abstract. We consider Clifford algebras in the context of Jackson calculus. We start by establishing the foundations for this q-deformed setting and introduce q-deformed partial derivatives

$$\partial_i^q f(x_1, \ldots, x_n) \;=\; \frac{f(x_1, \ldots, qx_i, \ldots, x_n) - f(x_1, \ldots, x_n)}{(q-1)\, x_i}\,.$$

The q-partial derivatives allow us to define operators for Clifford analysis such as the q-Dirac operator D_x^q, the q-Euler operator E^q, and the q-Gamma operator Γ^q:

$$D_x^q = \sum_{i=1}^{n} e_i\, \partial_i^q, \qquad E^q = \sum_{i=1}^{n} x_i\, \partial_i^q, \qquad \Gamma^q = \sum_{i<j} e_i e_j (x_i\, \partial_j^q - x_j\, \partial_i^q).$$

We then discuss relations between these operators.

This Jackson (or q-deformed) setting also leads to further peculiarities, which we investigate. In particular, the definition of q-analytic functions via a q-deformed Cauchy–Riemann operator

$$\bar{\partial} \;=\; \frac{1}{2}\left(\partial_x^q + i\, \partial_y^{1/q}\right)$$

yields interesting examples. With q-analytic functions we can also construct a q-Fock space and a q-Bargmann transform, which we illustrate here.