Weak solutions for time-fractional evolution equations

Daniela Sforza

Sapienza University of Roma (Italy)

School-Workshop on ANALYSIS, CONTROL & INVERSE PROBLEMS FOR DIFFUSIVE SYSTEMS WITH APPLICATION TO NATURAL AND SOCIAL SCIENCES

Bari, July 18-22, 2022

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1. abstract problems
2. fractional Petrovsky systems
3. boundary regularity

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$$\partial_t^\alpha u + Au = 0 \qquad\qquad \alpha \in (1,2)$$

- H Hilbert space $\quad \langle \cdot, \cdot \rangle$
- A densely defined linear self-adjoint positive operator on H

Loreti – S. Fractal Fract 2021

Riemann–Liouville integral operators

$$I^{\beta}u(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s)\, ds \qquad \beta > 0$$

$$\partial_t^{\alpha} u := I^{2-\alpha} u'' \qquad \alpha \in (1,2)$$

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Ω bounded open domain in $\mathbb{R}^N$ $(N \geq 1)$ smooth boundary $\partial\Omega$

$H = L^2(\Omega)$

1. $Au = -\Delta u \qquad D(A) = H^2(\Omega) \cap H^1_0(\Omega)$

2. $Au = \Delta^2 u \qquad D(A) = \{u \in H^4(\Omega) : u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0\}$

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$$\partial_t^\alpha u + Au = 0 \qquad \alpha \in (1, 2)$$

- $u \in C([0,T]; D(A)) \cap C^1([0,T]; H)$
- $\partial_t^\alpha u \in C([0,T]; H)$

$$\partial_t^\alpha u + Au = 0 \qquad \alpha \in (1,2)$$

- $u \in C([0,T]; D(A)) \cap C^1([0,T]; H)$
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$$\partial_t^\alpha u + Au = 0 \qquad \alpha \in (1,2)$$

- $u \in C([0,T]; D(A)) \cap C^1([0,T]; H)$
- $\partial_t^\alpha u \in C([0,T]; H)$
- u satisfies the equation for any $t \in [0,T]$

$$\partial_t^\alpha u + Au = 0 \qquad\qquad \alpha \in (1,2)$$

- $u \in C([0,T]; D(A)) \cap C^1([0,T]; H)$
- $\partial_t^\alpha u \in C([0,T]; H)$

$$\boxed{\partial_t^\alpha u = I^{2-\alpha} u'' = \frac{d}{dt} I^{2-\alpha}\big(u' - u'(0)\big)}$$

- $A^\theta \quad \theta > 0$
- $D(A^\theta)$ Hilbert space

$$\|u\|^2_{D(A^\theta)} := \sum_{n=1}^{\infty} \lambda_n^{2\theta} |\langle u, e_n\rangle|^2 \quad u \in D(A^\theta)$$

$$Ae_n = \lambda_n e_n$$

- $D(A^{-\theta}) := (D(A^\theta))'$ Hilbert space

$$\|\varphi\|^2_{D(A^{-\theta})} := \sum_{n=1}^{\infty} \lambda_n^{-2\theta} |\langle \varphi, e_n\rangle_{-\theta,\theta}|^2 \quad \varphi \in D(A^{-\theta})$$

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Definition

- $\forall v \in D(\sqrt{A}) \quad \langle I^{2-\alpha}\big(u' - u'(0)\big)(t), v\rangle \in C^1([0,T])$

$$\frac{d}{dt}\langle I^{2-\alpha}\big(u'-u'(0)\big)(t), v\rangle + \langle \sqrt{A}u(t), \sqrt{A}v\rangle = 0 \qquad t \in [0,T]$$

$$\partial_t^\alpha u + Au = 0 \qquad \alpha \in (1,2)$$

Definition

- $u \in C([0,T]; D(\sqrt{A}))$ $\qquad \sqrt{A} := A^{\frac{1}{2}}$
- $u' \in L^2(0,T;H) \cap C([0,T]; D(A^{-\theta})) \qquad \theta \in (0,1)$
- $\forall v \in D(\sqrt{A}) \quad \langle I^{2-\alpha}\big(u' - u'(0)\big)(t), v\rangle \in C^1([0,T])$

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$$\boxed{\partial_t^\alpha u = \frac{d}{dt} I^{2-\alpha}\big(u' - u'(0)\big)}$$

$$\partial_t^\alpha u + Au = 0 \qquad \text{in } (0,T)$$
$$u(0) = u_0 \quad u'(0) = u_1$$

$$u(t) = \sum_{n=1}^{\infty} \left[\langle u_0, e_n \rangle E_\alpha(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right] e_n$$

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- $Ae_n = \lambda_n e_n$
- Mittag–Leffler functions: $\alpha, \beta > 0 \quad z \in \mathbb{C}$

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \qquad E_\alpha(z) := E_{\alpha,1}(z)$$

$$u(t) = \sum_{n=1}^{\infty} \left[\langle u_0, e_n \rangle E_\alpha(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right] e_n$$

- scalar equations: $\alpha \in (1, 2)$

$$\partial_t^\alpha u + \lambda u = 0 \qquad \lambda > 0$$

$$u(0) = a \quad u'(0) = b \qquad a, b \in \mathbb{R}$$

$$u(t) = a\, E_\alpha(-\lambda t^\alpha) + b\, t E_{\alpha,2}(-\lambda t^\alpha)$$

1. $u_0 \in D(\sqrt{A})$ $u_1 \in H$ weak solution

$$u(t) = \sum_{n=1}^{\infty} \left[\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right] e_n$$

$u' \in C([0,T]; D(A^{-\theta}))$ $\theta \in \left(\frac{2-\alpha}{2\alpha}, \frac{1}{2}\right)$

2. $u_0 \in D(A)$ $u_1 \in D(\sqrt{A})$ strong solution

$$\partial_t^\alpha u(t)$$

$$= -\sum_{n=1}^{\infty} \lambda_n \left[\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right] e_n$$

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$$Au = \Delta^2 u \qquad D(A) = \{u \in H^4(\Omega) : u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0\}$$

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$$\begin{aligned} \partial_t^\alpha u + \Delta^2 u = 0 \qquad & \text{in } (0,T)\times\Omega \\ u = \Delta u = 0 \qquad & \text{on } (0,T)\times\partial\Omega \end{aligned}$$

u is a H^2-solution if

- $\forall v \in H^2(\Omega)\cap H_0^1(\Omega)$: $\int_\Omega I^{2-\alpha}\big(u_t - u_t(0)\big) v \, dx \in C^1([0,T])$

$$\frac{d}{dt}\int_\Omega I^{2-\alpha}\big(u_t - u_t(0)\big) v \, dx + \int_\Omega \Delta u \Delta v \, dx = 0 \qquad t \in [0,T]$$

$$\partial_t^\alpha u + \Delta^2 u = 0 \qquad \text{in } (0,T)\times\Omega$$
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u is a H^2-solution if

- $u \in C([0,T]; H^2(\Omega)\cap H_0^1(\Omega))$
- $u_t \in L^2(0,T;L^2(\Omega)) \cap C([0,T]; D((-\Delta)^{-\theta}))$ $\qquad \theta \in (0,1)$
- $\forall v \in H^2(\Omega)\cap H_0^1(\Omega) : \int_\Omega I^{2-\alpha}\big(u_t - u_t(0)\big) v\, dx \in C^1([0,T])$

$$\frac{d}{dt}\int_\Omega I^{2-\alpha}\big(u_t - u_t(0)\big) v\, dx + \int_\Omega \Delta u \Delta v\, dx = 0 \qquad t\in[0,T]$$

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weak solutions $\Longleftrightarrow$ H^2-solutions

$$
\begin{aligned}
\partial_t^\alpha u + \Delta^2 u &= 0 \qquad && \text{in } (0,T)\times\Omega \\
u = \Delta u &= 0 && \text{on } (0,T)\times\partial\Omega
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$$

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$$\begin{aligned} \Delta^2 u = f \qquad & \text{in } \Omega \\ u = \Delta u = 0 \qquad & \text{on } \partial\Omega \end{aligned}$$

Nazarov – Sweers 2007

u is a H^2-solution if

1. $u \in H^2(\Omega) \cap H^1_0(\Omega)$
2. $\forall v \in H^2(\Omega) \cap H^1_0(\Omega)$:

$$\int_\Omega (\Delta u \Delta v - fv)\, dx = 0$$

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$$\begin{aligned} \partial_t^\alpha u + \Delta^2 u = 0 &\qquad \text{in } (0,T)\times\Omega \\ u = \Delta u = 0 &\qquad \text{on } (0,T)\times\partial\Omega \end{aligned}$$

Definition

u is a H^3-solution if

- $\forall v \in H_0^1(\Omega)$

$$\int_\Omega \partial_t^\alpha u \, v \, dx - \int_\Omega \nabla\Delta u \cdot \nabla v \, dx = 0 \qquad t \in (0,T)$$

$$\begin{aligned} \partial_t^\alpha u + \Delta^2 u = 0 \qquad & \text{in } (0,T)\times\Omega \\ u = \Delta u = 0 \qquad & \text{on } (0,T)\times\partial\Omega \end{aligned}$$

Definition

u is a H^3-solution if

- $u \in C([0,T]; D(A^{\frac{3}{4}}))$

$$\boxed{D(A^{\frac{3}{4}}) = \{u \in H^3(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}}$$

- $\forall v \in H_0^1(\Omega)$

$$\int_\Omega \partial_t^\alpha u \, v \, dx - \int_\Omega \nabla\Delta u \cdot \nabla v \, dx = 0 \qquad t \in (0,T)$$

$$
\begin{aligned}
\partial_t^\alpha u + \Delta^2 u = 0 &\qquad \text{in } (0,T)\times\Omega \\
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\end{aligned}
$$

Definition

u is a H^3-solution if

- $u \in C([0,T]; D(A^{\frac{3}{4}}))$
- $u_t \in L^2(0,T; H^1(\Omega)) \cap C([0,T]; H^{-1}(\Omega))$
- $\partial_t^\alpha u \in L^2([0,T]; L^2(\Omega))$
- $\forall v \in H_0^1(\Omega)$

$$\int_\Omega \partial_t^\alpha u \, v \, dx - \int_\Omega \nabla\Delta u \cdot \nabla v \, dx = 0 \qquad t \in (0,T)$$

$$\begin{aligned} \partial_t^\alpha u + \Delta^2 u = 0 \qquad & \text{in } (0,T)\times\Omega \\ u = \Delta u = 0 \qquad & \text{on } (0,T)\times\partial\Omega \end{aligned}$$

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- $\forall v \in H_0^1(\Omega)$

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- $u_0 \in D(A^{\frac{3}{4}}) = \{u \in H^3(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}$
  $u_1 \in H_0^1(\Omega)$

$$u(t) = \sum_{n=1}^{\infty} \left[\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right] e_n$$

H^3-solution

- $u_0 \in D(A^{\frac{3}{4}}) = \{u \in H^3(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}$
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H^3-solution

- for some $C > 0$

$$\|\partial_t^\alpha u\|_{L^2(0,T;L^2(\Omega))} \leq C\left(\|\nabla \Delta u_0\|_{L^2(\Omega)} + \|\nabla u_1\|_{L^2(\Omega)}\right)$$

H^3 existence and regularity

- $u_0 \in D(A^{\frac{3}{4}}) = \{u \in H^3(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}$
  $u_1 \in H_0^1(\Omega)$

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H^3-solution

- $\theta \in \left(0, \frac{1}{2\alpha}\right)$

$$\|\partial_t^\alpha u\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\Delta u\|_{L^2(0,T;D(A^\theta))} \leq C\big(\|\nabla\Delta u_0\|_{L^2(\Omega)} + \|\nabla u_1\|_{L^2(\Omega)}\big)$$

$$\begin{aligned} \partial_t^\alpha u + \Delta^2 u &= 0 && \text{in } (0,T)\times\Omega \\ u = \Delta u &= 0 && \text{on } (0,T)\times\partial\Omega \end{aligned}$$

$$u \in C([0,T]; H_0^1(\Omega)) \ H^1\text{-solution} \overset{\text{Definition}}{\Longleftrightarrow} (-\Delta)^{-1}u \ H^3\text{-solution}$$

$$\partial_t^\alpha u + \Delta^2 u = 0 \qquad \text{in } (0,T)\times\Omega$$
$$u = \Delta u = 0 \qquad \text{on } (0,T)\times\partial\Omega$$

$u \in C([0,T]; H_0^1(\Omega))$ H^1-solution $\overset{\text{Definition}}{\Longleftrightarrow}$ $(-\Delta)^{-1}u$ H^3-solution

H^1 existence:

$u_0 \in H_0^1(\Omega) \quad u_1 \in H^{-1}(\Omega)$

$$u(t,x) = \sum_{n=1}^{\infty} \left[\langle u_0, e_n\rangle E_\alpha(-\lambda_n t^\alpha) + \langle u_1, e_n\rangle_{H^{-1}} t E_{\alpha,2}(-\lambda_n t^\alpha)\right] e_n(x)$$

H^1-solution

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$$\begin{aligned} &\partial_t^2 u + \triangle^2 u = 0 && (0,T)\times\Omega \\ &u = \Delta u = 0 && (0,T)\times\partial\Omega \end{aligned}$$

Lasiecka – Triggiani 1983-1990 J.-L. Lions 1983

$$\int_0^T \int_{\partial\Omega} |\partial_\nu u|^2 \, d\sigma \, dt \le C\big(\|u(0)\|^2_{H_0^1(\Omega)} + \|u_t(0)\|^2_{H^{-1}(\Omega)}\big)$$

$$\begin{aligned} &\partial_t^\alpha u + \Delta^2 u = 0 && \text{in } (0,T)\times\Omega \\ &u = \Delta u = 0 && \text{on } (0,T)\times\partial\Omega \\ &u(0) = u_0 \quad u_t(0) = u_1 \end{aligned}$$

Loreti – S. accepted FCAA

$u_0 \in H_0^1(\Omega)$ $u_1 \in H^{-1}(\Omega)$

$$\int_0^T \int_{\partial\Omega} |\partial_\nu u|^2 \, d\sigma \, dt \le C\left(\|u_0\|^2_{H_0^1(\Omega)} + \|u_1\|^2_{H^{-1}(\Omega)}\right)$$

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$$\begin{aligned}
&\partial_t^\alpha u + \Delta^2 u = 0 && \text{in } (0,T)\times\Omega \\
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$$\begin{cases} \partial_t^\alpha u + \Delta^2 u = 0 & \text{in } (0,T)\times\Omega \\ u = \Delta u = 0 & \text{on } (0,T)\times\partial\Omega \\ u(0) = u_0 \quad u_t(0) = u_1 & \end{cases}$$

Theorem

$u_0 \in \{u \in H^4(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}$
$u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$

$$\int_0^T \int_{\partial\Omega} |\partial_\nu \Delta u|^2 d\sigma dt \leq C\big(\|\nabla\Delta u_0\|^2_{L^2(\Omega)} + \|\nabla u_1\|^2_{L^2(\Omega)}\big)$$

adapting Komornik 1994

boundary regularity for strong solutions

$$
\begin{aligned}
&\partial_t^\alpha u + \Delta^2 u = 0 && \text{in } (0,T)\times\Omega \\
&u = \Delta u = 0 && \text{on } (0,T)\times\partial\Omega \\
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\end{aligned}
$$

Theorem

$u_0 \in \{u \in H^4(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}$
$u_1 \in H^2(\Omega) \cap H^1_0(\Omega)$

$$
\int_0^T \int_{\partial\Omega} \left|\partial_\nu \Delta u\right|^2 d\sigma dt \le C\left(\|\nabla\Delta u_0\|^2_{L^2(\Omega)} + \|\nabla u_1\|^2_{L^2(\Omega)}\right)
$$

adapting Komornik 1994

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- $\beta \in (0,1) \quad T > 0 \quad (H, \|\cdot\|_H)$ Hilbert space

$$u \in H^\beta(0,T;H)$$

$$\Updownarrow$$

$$u \in L^2(0,T;H) \quad \& \quad \frac{\|u(t)-u(\tau)\|_H}{|t-\tau|^{\frac{1}{2}+\beta}} \in L^2([0,T]\times[0,T])$$

$$\|u\|^2_{H^\beta(0,T;H)} := \int_0^T \|u(t)\|^2_H dt + \int_0^T \int_0^T \frac{\|u(t)-u(\tau)\|^2_H}{|t-\tau|^{1+2\beta}} dt d\tau$$

- extension of a result in Gorenflo-Luchko-Yamamoto 2015

- $\beta \in (0,1) \quad T>0 \quad (H, \|\cdot\|_H)$ Hilbert space

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- extension of a result in Gorenflo-Luchko-Yamamoto 2015

$$\|I^\beta u\|_{H^\beta(0,T;H)} \sim \|u\|_{L^2(0,T;H)}$$

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau)\, d\tau \qquad \beta > 0$$

Riemann–Liouville integral operators

- $\beta \in (0,1) \quad T > 0 \quad (H, \|\cdot\|_H)$ Hilbert space

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- extension of a result in Gorenflo-Luchko-Yamamoto 2015

$$\|I^{\beta} u\|_{H^{\beta}(0,T;H)} \sim \|u\|_{L^2(0,T;H)}$$

- $\beta \in (0,1) \quad T > 0 \quad (H, \|\cdot\|_H)$ Hilbert space

$$\|u\|^2_{H^\beta(0,T;H)} := \int_0^T \|u(t)\|^2_H dt + \int_0^T \int_0^T \frac{\|u(t) - u(\tau)\|^2_H}{|t-\tau|^{1+2\beta}} dt d\tau$$

- extension of a result in Gorenflo-Luchko-Yamamoto 2015

$$\|I^\beta u\|_{H^\beta(0,T;H)} \sim \|u\|_{L^2(0,T;H)}$$

$$\|\partial_\nu \Delta u\|_{L^2(0,T;L^2(\partial\Omega))} \sim \|I^\beta(\partial_\nu \Delta u)\|_{H^\beta(0,T;L^2(\partial\Omega))}$$

$$\|\partial_\nu \Delta u\|_{L^2(0,T;L^2(\partial\Omega))} \sim \|I^\beta(\partial_\nu \Delta u)\|_{H^\beta(0,T;L^2(\partial\Omega))}$$

- by the multiplier method

$$\|I^\beta(\partial_\nu \Delta u)\|_{H^\beta(0,T;L^2(\partial\Omega))} \le C\Big(\|I^\beta(\partial_t^\alpha u)\|_{H^\beta(0,T;L^2(\Omega))} + \|I^\beta(\nabla \Delta u)\|_{H^\beta(0,T;L^2(\Omega))}\Big)$$

- $\|I^\beta(\partial_t^\alpha u)\|_{H^\beta(0,T;L^2(\Omega))} \sim \|\partial_t^\alpha u\|_{L^2(0,T;L^2(\Omega))}$

  $\|I^\beta(\nabla \Delta u)\|_{H^\beta(0,T;L^2(\Omega))} \sim \|\nabla \Delta u\|_{L^2(0,T;L^2(\Omega))}$

$$\|\partial_\nu \Delta u\|_{L^2(0,T;L^2(\partial\Omega))} \sim \|I^\beta(\partial_\nu \Delta u)\|_{H^\beta(0,T;L^2(\partial\Omega))}$$

- by the multiplier method

$$\left\|I^\beta(\partial_\nu \Delta u)\right\|_{H^\beta(0,T;L^2(\partial\Omega))}$$
$$\leq C\left(\left\|I^\beta(\partial_t^\alpha u)\right\|_{H^\beta(0,T;L^2(\Omega))} + \left\|I^\beta(\nabla \Delta u)\right\|_{H^\beta(0,T;L^2(\Omega))}\right)$$

- $\left\|I^\beta(\partial_t^\alpha u)\right\|_{H^\beta(0,T;L^2(\Omega))} \sim \|\partial_t^\alpha u\|_{L^2(0,T;L^2(\Omega))}$

$$\left\|I^\beta(\nabla \Delta u)\right\|_{H^\beta(0,T;L^2(\Omega))} \sim \|\nabla \Delta u\|_{L^2(0,T;L^2(\Omega))}$$

$$\|\partial_\nu \Delta u\|_{L^2(0,T;L^2(\partial\Omega))} \sim \|I^\beta(\partial_\nu \Delta u)\|_{H^\beta(0,T;L^2(\partial\Omega))}$$

- by the multiplier method $\langle \cdot, \cdot \rangle$ $h \cdot \nabla \Delta I^\beta(u)$

$$\left\|I^\beta(\partial_\nu \Delta u)\right\|_{H^\beta(0,T;L^2(\partial\Omega))}$$
$$\leq C\left(\left\|I^\beta(\partial_t^\alpha u)\right\|_{H^\beta(0,T;L^2(\Omega))} + \left\|I^\beta(\nabla \Delta u)\right\|_{H^\beta(0,T;L^2(\Omega))}\right)$$

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$$\left\|I^\beta(\nabla \Delta u)\right\|_{H^\beta(0,T;L^2(\Omega))} \sim \|\nabla \Delta u\|_{L^2(0,T;L^2(\Omega))}$$

$$\|\partial_\nu \Delta u\|_{L^2(0,T;L^2(\partial\Omega))} \sim \|I^\beta(\partial_\nu \Delta u)\|_{H^\beta(0,T;L^2(\partial\Omega))}$$

- by the multiplier method $\partial_t^\alpha I^\beta(u) \neq I^\beta(\partial_t^\alpha u)$

$$\left\|I^\beta(\partial_\nu \Delta u)\right\|_{H^\beta(0,T;L^2(\partial\Omega))} \leq C\Big(\left\|I^\beta(\partial_t^\alpha u)\right\|_{H^\beta(0,T;L^2(\Omega))} + \left\|I^\beta(\nabla \Delta u)\right\|_{H^\beta(0,T;L^2(\Omega))}\Big)$$

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$$\|\partial_\nu \Delta u\|_{L^2(0,T;L^2(\partial\Omega))} \sim \|I^\beta(\partial_\nu \Delta u)\|_{H^\beta(0,T;L^2(\partial\Omega))}$$

- by the multiplier method

$$\begin{aligned}&\left\|I^\beta(\partial_\nu \Delta u)\right\|_{H^\beta(0,T;L^2(\partial\Omega))}\\ &\le C\Big(\left\|I^\beta(\partial_t^\alpha u)\right\|_{H^\beta(0,T;L^2(\Omega))} + \left\|I^\beta(\nabla \Delta u)\right\|_{H^\beta(0,T;L^2(\Omega))}\Big)\end{aligned}$$

- $\left\|I^\beta(\partial_t^\alpha u)\right\|_{H^\beta(0,T;L^2(\Omega))} \sim \|\partial_t^\alpha u\|_{L^2(0,T;L^2(\Omega))}$

$$\left\|I^\beta(\nabla \Delta u)\right\|_{H^\beta(0,T;L^2(\Omega))} \sim \|\nabla \Delta u\|_{L^2(0,T;L^2(\Omega))}$$

$$\|\partial_t^\alpha u\|_{L^2(0,T;L^2(\Omega))} + \|\nabla \Delta u\|_{L^2(0,T;D(A^\theta))} \leq C\left(\|\nabla \Delta u_0\|_{L^2(\Omega)} + \|\nabla u_1\|_{L^2(\Omega)}\right)$$

$$\theta \in \left(0, \tfrac{1}{2\alpha}\right)$$

strong solutions $\Longrightarrow$ H^3-solutions

- u strong solution

$$\begin{aligned}\|\partial_\nu \Delta u\|_{L^2(0,T;L^2(\partial\Omega))} &\sim \left\|I^\beta(\partial_\nu \Delta u)\right\|_{H^\beta(0,T;L^2(\partial\Omega))} \\ &\leq C\big(\|\partial_t^\alpha u\|_{L^2(0,T;L^2(\Omega))} + \|\nabla \Delta u\|_{L^2(0,T;L^2(\Omega))}\big) \\ &\leq C\big(\|\nabla \Delta u_0\|_{L^2(\Omega)} + \|\nabla u_1\|_{L^2(\Omega)}\big)\end{aligned}$$

- u H^2-solution

$$\|\partial_\nu u\|_{L^2(0,T;L^2(\partial\Omega))} \leq C\big(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{H^{-1}(\Omega)}\big)$$

- u H^1-solution

$$\|\partial_\nu u\|_{L^2(0,T;L^2(\partial\Omega))} \leq C\big(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{H^{-1}(\Omega)}\big)$$

- u strong solution

$$\|\partial_\nu \Delta u\|_{L^2(0,T;L^2(\partial\Omega))} \leq C\big(\|\nabla\Delta u_0\|_{L^2(\Omega)} + \|\nabla u_1\|_{L^2(\Omega)}\big)$$

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$$\|\partial_\nu \Delta u\|_{L^2(0,T;L^2(\partial\Omega))} \leq C\big(\|\nabla \Delta u_0\|_{L^2(\Omega)} + \|\nabla u_1\|_{L^2(\Omega)}\big)$$

- u H^2-solution $\Longrightarrow (-\Delta)^{-1}u$ strong solution

$$\begin{aligned}&\|\partial_\nu \Delta(-\Delta)^{-1} u\|_{L^2(0,T;L^2(\partial\Omega))}\\ &\qquad \leq C\big(\|\nabla \Delta(-\Delta)^{-1} u_0\|_{L^2(\Omega)} + \|\nabla(-\Delta)^{-1} u_1\|_{L^2(\Omega)}\big)\end{aligned}$$

- u H^2-solution

$$\|\partial_\nu u\|_{L^2(0,T;L^2(\partial\Omega))} \leq C\big(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{H^{-1}(\Omega)}\big)$$

- u H^1-solution

$$\|\partial_\nu u\|_{L^2(0,T;L^2(\partial\Omega))} \leq C\big(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{H^{-1}(\Omega)}\big)$$

trace regularity: conclusions

- u strong solution

$$\|\partial_\nu \Delta u\|_{L^2(0,T;L^2(\partial\Omega))} \le C\left(\|\nabla\Delta u_0\|_{L^2(\Omega)} + \|\nabla u_1\|_{L^2(\Omega)}\right)$$

- u H^2-solution

$$\|\partial_\nu u\|_{L^2(0,T;L^2(\partial\Omega))} \le C\left(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{H^{-1}(\Omega)}\right)$$

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- u H^1-solution

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1. Dirichlet–Neumann boundary conditions

$$\begin{cases} \partial_t^\alpha u + \Delta^2 u = 0 & \text{in } (0,T) \times \Omega \\ u = \partial_\nu u = 0 & \text{on } (0,T) \times \partial\Omega \end{cases}$$

2. polyharmonic operator Δ^{2m} of order $2m$

$$\begin{cases} \partial_t^\alpha u + \Delta^{2m} u = 0 & \text{in } (0,T) \times \Omega \\ u = \Delta u = \ldots = \Delta^{2m-1} u = 0 & \text{on } (0,T) \times \partial\Omega \end{cases}$$

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