Recent results on the Hegselmann-Krause opinion formation models with time-dependent time delays

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Bari, July 18-22, 2022
Analysis, Control & Inverse Problems for Diffusive Systems with Application to Natural and Social Sciences.

Joint work with Cristina Pignotti (University of L'Aquila)

The undelayed HK model

- Consider N agents. Each opinion can be represented as $x_i : [0, +\infty) \to \mathbb{R}^d$.
- The dynamics of the N opinions follows the continuum Hegselmann-Krause model:

$$\frac{dx_i(t)}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \psi_{ij}(t)(x_j(t) - x_i(t)), \quad i = 1, \ldots, N, \qquad (1.1)$$
$$x_i(0) = x_{i,0}.$$

R. Hegselmann and U. Krause, *Opinion dynamics and bounded confidence, models, analysis and simulation*, (2002).

- $\psi_{ij}(t) := \psi(|x_j(t) - x_i(t)|)$ for any $i, j = 1, \ldots, N$, where
  - $\psi_{ij}(t) \geq 0$;
  - ψ_{ij} is non-increasing;
  - $\psi(0) = 1$.
- $\lambda > 0$ is the coupling strenght.

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  - $\psi_{ij}(t) \geq 0$;
  - ψ_{ij} is non-increasing;
  - $\psi(0) = 1$.
- $\lambda > 0$ is the coupling strenght.

We say that the solution to (1.1) converges to **consensus** if for any $i, j \in \{1, \dots, N\}$

$$\lim_{t \to +\infty} |x_i(t) - x_j(t)| = 0.$$

The undelayed model: References

Consensus for the Hegselmann-Krause without time delay has been studied in the following papers:

- S. Mcquade, B. Piccoli, N. Pouradier Duteil *Social Dynamics Models with Time-Varying Influence*, 2019;
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Possible extension:

- Hegselmann-Krause model and control theory:
  - S. Wongkaew, M. Caponigro, A. Borzi, *On the control through leadership of the Hegselmann Krause opinion formation model*, 2015:

$$\dot{x}_0 = u(t),$$
$$\dot{x}_i = \sum_{j=1}^{N} \psi_{ij}(x_j - x_i) + \gamma\phi(|x_0 - x_i|)(x_0 - x_i).$$

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  - B. Piccoli, N. Pouradier Duteil, E. Trelat, *Sparse control of Hegselmann-Krause models: Black hole and declustering*, 2018:

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} \psi_{ij}(x_j - x_i) + u_i(t).$$

Extension

- H-K model with bounded confidence and negative interaction:
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- Kinetic model:
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- Compactly supported potential: $supp\ \psi_{ij}(r) \subset [0, 1]$, for any $i, j \in \{1, \dots, N\}$:

$$\frac{dx_i(t)}{dt} = \frac{1}{\sum_{k=1}^N \psi_{ik}(t)} \sum_{j=1}^N \psi_{ij}(t)(x_j(t) - x_i(t)). \tag{1.2}$$

  - Cluster phenomenon.
  - P.-E. Jabin, S. Motsch, *Clustering and asymptotic behavior in opinion formation*, 2014.
- Delay feedback
  - Y.-P. Choi, A. P., C. Pignotti, Consensus of the Hegselmann–Krause opinion formation model with time delay, *Math. Meth Appl. Sci.*, 44(6), 2021, 4560-4579.
  - A. P., Convergence to consensus for a Hegselmann-Krause-type model with distributed time delay, *Minimax Theory and its Application*. 6 (2021), n.2, pp. 379-394.
  - J. Haskovec Direct proof of unconditional asymptotic consensus in the Hegselmann-Krause model with transmission-type delay, Bull. Lond. Math. Soc., 53, (2021), pp. 13121323.
  - E. Continelli, C. Pignotti, Consensus for the Hegselmann-Krause model with time variable time delays, Preprint 2022.

Let us consider N agents, $N \in \mathbb{N}$, and let us denote by $x_i(\cdot) : [0, +\infty) \to \mathbb{R}^d$ the opinion of the i-th agent, for any $i \in \{1, ..., N\}$, which obeys to the following Hegselmann-Krause-type model

$$\begin{aligned} \frac{dx_i}{dt}(t) &= \frac{1}{N} \sum_{j \neq i} a_{ij}(t)(x_j(t - \tau(t)) - x_i(t)), \quad t \geq 0, \\ x_i(s) &= x_{i,0}(s), \quad s \in [-\bar{\tau}, 0], \end{aligned} \tag{2.3}$$

where $\tau(\cdot) : [0, +\infty) \to [0, +\infty)$ is the time-dependent time delay, which is a continuous function satisfying

$$0 \leq \tau(t) \leq \bar{\tau}, \qquad t \geq 0. \tag{2.4}$$

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Moreover, for any $i, j \in \{1, ..., N\}$, the influence rates are defined as

$$a_{ij}(t) := a(x_j(t-\tau(t)), x_i(t)), \quad \forall\, t \geq 0, \tag{2.5}$$

with $a : \mathbb{R}^d \times \mathbb{R}^d \to (0, +\infty)$ continuous and locally uniformly Lipschitz continuous in the second argument. Without loss of generality, we may assume

$$\|a\|_\infty \leq 1. \tag{2.6}$$

Then, in particular,

$$\frac{1}{N}\sum_{j\neq i} a_{ij}(t) \leq 1, \quad \forall\, i \in \{1, \dots, N\}. \tag{2.7}$$

Furthermore, we suppose that there exists a non-increasing continuous function $\psi : [0, +\infty) \to (0, +\infty)$ such that

$$a(x, y) \geq \psi(|x-y|) > 0, \quad \forall x, y \in \mathbb{R}^d. \tag{2.8}$$

Finally, we take initial data $x_{i,0}(\cdot) \in \mathcal{C}([-\bar{\tau}, 0]),\ \forall\, i = 1, \dots, N.$

The particular choices studied in Choi, P., Pignotti (2021)

$$a_{ij} = \begin{cases} \psi_{ij}(t) & \rightarrow \quad \textbf{(symmetric case)}, \\ \frac{N\psi_{ij}(t)}{\sum_{k=1}^{N} \psi_{ik}(t)} & \rightarrow \quad \textbf{(non-symmetric case)} \end{cases}$$

fall in this framework, but with the further assumption: $\tau'(t) \leq c < 1$.

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We are interested in studying the consensus behavior of the solution to (2.3). For this, let us introduce the diameter functional as

$$d_X(t) := \max_{i,j\in\{1,\ldots,N\}} |x_i(t) - x_j(t)|, \quad t \geq -\bar{\tau}.$$

Then, the following definition holds.

Definition

We say that the solution $\{x_i\}_{i\in\{1,\ldots,N\}}$ to (2.3) converges to *consensus* if and only if

$$\lim_{t\to+\infty} d_X(t) = 0.$$

Lemma

Let $\{x_i\}_{i\in\{1,\dots,N\}}$ *be the solution to* (2.3). *Then, for any fixed* $\bar{x} \in \mathbb{R}^d$,

$$\min_{j\in\{1,\dots,N\}} \min_{s\in[-\bar{\tau},0]} \langle x_j(s), \bar{x}\rangle \le \langle x_i(t), \bar{x}\rangle \le \max_{j\in\{1,\dots,N\}} \max_{s\in[-\bar{\tau},0]} \langle x_j(s), \bar{x}\rangle, \tag{2.9}$$

for any $t \ge -\bar{\tau}$ *and* $i \in \{1, ..., N\}$.

Lemma

Let $\{x_i\}_{i\in\{1,\dots,N\}}$ *be the solution to* (2.3) *and let us denote*

$$D_0 := \max_{s,t\in[-\bar{\tau},0]} \max_{i,j\in\{1,\dots,N\}} |x_i(s) - x_j(t)|. \tag{2.10}$$

Then, for all $i, j \in \{1, \dots, N\}$,

$$|x_j(s) - x_i(t)| \le D_0, \quad \forall\, t, s \ge -\bar{\tau}. \tag{2.11}$$

Lemma 15 gives a positive lower bound for the influence weights a_{ij}. Indeed, for any $i, j \in \{1, ..., N\}$ and $t \geq 0$,

$$a_{ij}(t) \geq \psi(|x_j(t - \tau(t)) - x_i(t)|) \geq \psi(D_0) > 0. \tag{2.12}$$

Since the functional $d_X(t)$ may be not differentiable for some $t > 0$, then in the following computations we will use the upper Dini derivate D^+ defined as

$$D^+F(t) := \limsup_{h \to 0+} \frac{F(t+h) - F(t)}{h},$$

for any continuous function F. Hence, the following estimate on d_X holds.

Lemma

Let $\{x_i\}_{i \in \{1,..,N\}}$ be the solution to (2.3). If we define

$$\sigma_\tau(t) := \int_{t-\bar{\tau}}^{t} \max_{k \in \{1,\dots,N\}} \left| \frac{dx_k}{ds}(s) \right| ds, \qquad \forall\, t \geq \bar{\tau}, \tag{2.13}$$

then, we get

$$D^+ d_X(t) \leq 2\sigma_\tau(t) - \psi(D_0) d_X(t), \tag{2.14}$$

for any $t \geq \bar{\tau}$.

Lemma

Let $\{x_i\}_{i\in\{1,\dots,N\}}$ *be the solution to* (2.3). *Then,*

$$\max_{i\in\{1,\dots,N\}}\left|\frac{dx_i}{dt}(t)\right| \leq \sigma_\tau(t)+d_X(t), \tag{2.15}$$

for any $t \geq \bar{\tau}$.

Theorem (A. P., C. Pignotti (2022))

Let $\{x_i\}_{i\in\{1,\dots,N\}}$ *be the solution to* (2.3). *If*

$$\bar{\tau} < \ln\left(1+\frac{\psi(D_0)}{2+\psi(D_0)}\right), \tag{2.16}$$

then, there exist $C, \tilde{C} > 0$ *independent of* N *such that*

$$d_X(t) \leq \tilde{C}e^{-Ct}, \tag{2.17}$$

for any $t \geq 0$. *Hence, consensus occurs exponentially.*

Let $\mathcal{M}(\mathbb{R}^d)$ be the set of probability measures on the space $\mathbb{R}^d$. Then, the continuum model associated to the particle system (2.3) is given by

$$\begin{aligned} &\partial_t \mu_t + \operatorname{div}(F[\mu_{t-\tau(t)}]\mu_t) = 0, \quad t > 0, \\ &\mu_s =: g_s, \quad x \in \mathbb{R}^d, \quad s \in [-\bar{\tau}, 0], \end{aligned} \tag{2.18}$$

where the velocity field F is given by

$$F[\mu_{t-\tau(t)}](x) = \int_{\mathbb{R}^d} a(y,x)(y-x)\, d\mu_{t-\tau(t)}(y), \tag{2.19}$$

and $g_s \in \mathcal{C}([-\tau(0), 0]; \mathcal{M}(\mathbb{R}^d))$. We assume that the delay function $\tau(\cdot)$ is bounded from below by a positive constant, namely there exists $\tau^* > 0$ such that

$$\tau(t) \geq \tau^*, \quad \forall\, t \geq 0.$$

Moreover, we assume that the potential $a(\cdot,\cdot)$ in (2.5) is Lipschitz continuous in both arguments, namely for any $(x,y),(x',y') \in \mathbb{R}^{2d}$ there exists $L > 0$ such that

$$|a(y,x) - a(y',x')| \leq L(|y-y'| + |x-x'|).$$

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$$|a(y,x) - a(y',x')| \leq L(|y-y'| + |x-x'|).$$

Definition

Let $T > 0$. We say that $\mu_t \in \mathcal{C}([0,T); \mathcal{M}(\mathbb{R}^d))$ is a measure-valued solution to (2.18) on the time interval $[0,T)$ if for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times [0,T))$ we have:

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t \varphi + F[\mu_{t-\tau(t)}](x) \cdot \nabla_x \varphi\right) d\mu_t(x) dt + \int_{\mathbb{R}^d} \varphi(x,0) dg_0(x) = 0. \tag{2.20}$$

Lemma

Let $\mu_t \in \mathcal{C}([0,T];\mathcal{M}(\mathbb{R}^d))$ *have uniform compact support, i.e. there exist some* $R > 0$ *such that*

$$supp\ \mu_t \subset B^d(0,R), \quad \forall\, t \in [0,T].$$

Then there exists a constant $K > 0$ *such that for all* $x, \tilde{x} \in B^d(0,R)$ *and for all* $t \in [0,T]$

$$|F[\mu_{t-\tau(t)}](x) - F[\mu_{t-\tau(t)}](\tilde{x})| \leq C|x - \tilde{x}|. \tag{2.21}$$

Moreover, there exists a constant $K > 0$ *such that for all* $x \in B^d(0,R)$ *and for all* $t \in [0,T]$*.*

$$|F[\mu_{t-\tau(t)}](x)| \leq K. \tag{2.22}$$

Theorem

Consider the system (2.18) *with* $g_s \in \mathcal{C}([-\bar{\tau},0];\mathcal{P}_1(\mathbb{R}^d))$. *Suppose that there exists a constant* $R > 0$ *such that* $supp\ g_t \in B^d(0,R)$, *for all* $t \in [-\bar{\tau},0]$, *where* $B^d(0,R)$ *denotes the ball of radius* R *in* $\mathbb{R}^d$ *centered at the origin. Then, for any* $T > 0$ *there exists a unique measure-valued solution* $\mu_t \in \mathcal{C}([0,T);\mathcal{P}_1(\mathbb{R}^d))$ *of* (2.18) *in the sense of* (2.20)*. Moreover,* μ_t *is uniformly compactly supported and*

$$\mu_t = X(t;\cdot)\#\mu_0, \tag{2.23}$$

where $X(t;\cdot)$ *is the solution to the characteristic system associated to* (2.18) *for any* $t \in [0,T)$*.*

Lemma

Let $\mu_t^1, \mu_t^2 \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d))$ *be two measure-valued solutions to* (2.18)*, with compactly supported initial data* $g_s^1, g_s^2 \in \mathcal{C}([-\bar{\tau},0];\mathcal{P}_1(\mathbb{R}^d))$ *respectively. Then, there exists a constant* $C > 0$ *depending only on* T *such that*

$$d_1(\mu_t^1, \mu_t^2) \leq C \max_{s\in[-\bar{\tau},0]} d_1(g_s^1, g_s^2), \tag{2.24}$$

for any $t \in [0,T]$*.*

Theorem

Let $\mu_t \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d))$ *be a measure-valued solution to* (2.18) *with compactly supported initial datum* $g_s \in \mathcal{C}([-\bar{\tau},0];\mathcal{P}_1(\mathbb{R}^d))$ *and let* F *as in* (2.19)*. Suppose that*

$$\bar{\tau} < \ln\left(1 + \frac{\psi\left(\max_{s\in[-\bar{\tau},0]} d_X(g_s)\right)}{2+\psi\left(\max_{s\in[-\bar{\tau},0]} d_X(g_s)\right)}\right). \tag{2.25}$$

Then, there exists a constant $C > 0$ *such that*

$$d_X(\mu_t) \leq \left(\max_{s\in[-\bar{\tau},0]} d_X(g_s)\right) e^{-Ct}, \tag{2.26}$$

for all $t \geq 0$*.*

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- Let $N \in \mathbb{N}$ be the number of agents.
- Let $x_i : [0, +\infty) \to \mathbb{R}^d$ be the opinion of the i-th agent, for any $i = 1, \dots, N$.
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Then, the dynamics is governed by the following modified Hegselmann-Krause model with leadership:

$$\begin{aligned} \frac{dx_0}{dt}(t) &= u(t), \\ \frac{dx_i}{dt}(t) &= \frac{1}{N} \sum_{j \neq i} a_{ij}(t)(x_j(t - \tau(t)) - x_i(t)) \\ &\quad + \gamma \phi(|x_0(t - \tau(t)) - x_i(t)|)(x_0(t - \tau(t)) - x_i(t)), \end{aligned} \tag{3.27}$$

for all $t \geq 0$, with continuous initial data, for $i = 1, \dots, N$,

$$x_0(s) = x_{0,0}(s), \quad x_i(s) = x_{i,0}(s), \quad \forall\, s \in [-\bar{\tau}, 0]. \tag{3.28}$$

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- $u : [0, +\infty) \to \mathbb{R}^d$ is a measurable function, which represents the result of the leader's opinion.

Definition

We say that a measurable control $t \mapsto u(t)$ is **admissible** if there exists a constant $M > 0$ such that $||u||_{L^\infty} \leq M$.

- $\tau(\cdot) : [0, +\infty) \to [0, +\infty)$ is the time-dependent time delay. We assume that $\tau(\cdot)$ is continuous and bounded, i.e.
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- The influence functions a_{ij} are of the form
$$a_{ij}(t) = a(|x_j(t - \tau(t)) - x_i(t)|),$$
where $a : [0, +\infty) \to [0, +\infty)$ is a continuous non-increasing cut-off function.

  In particular, we assume:
$$a(s) = 1 \quad \text{for } s \in [0, \delta], \quad a(s) = 0 \quad \text{for } s \geq r,$$
where $\delta, r \in \mathbb{R}^+$, with $r > \delta$.

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- $\gamma > 0$ represents the strength of the leader's opinion.
- $\phi(\cdot) : [0, +\infty) \to \mathbb{R}^+$ is the leader's influence function on the agents.

We assume:

  - ϕ non-increasing and Lipschitz continuous, with Lipschitz constant $L > 0$.

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  - ϕ non-increasing and Lipschitz continuous, with Lipschitz constant $L > 0$.
  - Without loss of generality, we can take $\phi(0) = 1$.

- $\tau(\cdot) : [0, +\infty) \to [0, +\infty)$ is the time-dependent time delay. We assume that $\tau(\cdot)$ is continuous and bounded, i.e.

$$\tau(t) \leq \bar{\tau}, \quad t \geq 0.$$

- The influence functions a_{ij} are of the form

$$a_{ij}(t) = a(|x_j(t - \tau(t)) - x_i(t)|),$$

where $a : [0, +\infty) \to [0, +\infty)$ is a continuous non-increasing cut-off function.

In particular, we assume:

$$a(s) = 1 \quad \text{for } s \in [0, \delta], \quad a(s) = 0 \quad \text{for } s \geq r,$$

where $\delta, r \in \mathbb{R}^+$, with $r > \delta$.

- $\gamma > 0$ represents the strength of the leader's opinion.
- $\phi(\cdot) : [0, +\infty) \to \mathbb{R}^+$ is the leader's influence function on the agents.

  We assume:
  - ϕ non-increasing and Lipschitz continuous, with Lipschitz constant $L > 0$.
  - Without loss of generality, we can take $\phi(0) = 1$.
- If $\tau(t) \equiv 0 \ \to$ S. Wongkaew, M. Caponigro, A. Borzi, *On the control through leadership of the Hegselmann–Krause opinion formation model*, (2015).

First step: to study the consensus behavior of solution in the sense of the following definition.

Definition

Let $\{x_i\}_{i=0,1,\dots,N}$ be a solution to (3.27)-(3.28). We say that the solution converges to **consensus** if and only if

$$\lim_{t\to+\infty} |x_i(t) - x_j(t)| = 0,$$

for any $i, j \in \{0, 1, \dots, N\}$.

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To do so, we define the maximal distance from the leader's opinion:

$$d_0(t) := \max_{i\in\{1,\dots,N\}} |x_i(t) - x_0(t)|, \quad t \geq -\bar{\tau},$$

and let $p = p(t) \in \{1, ..., N\}$ be such that

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Without loss of generality, we can take $p = 1$. Consider the control

$$u(t) := \gamma\alpha(t) \sum_{j=1}^{N} \phi(|x_j(t-\tau(t)) - x_0(t)|)(x_j(t-\tau(t)) - x_0(t)), \tag{3.29}$$

where

$$\alpha(t) := \frac{1}{2} \min\left\{ \frac{\phi(|x_1(t-\tau(t)) - x_0(t)|)}{N}, \frac{2M}{\gamma \sum_{j=1}^{N} |x_j(t-\tau(t)) - x_0(t)|} \right\}.$$

- u in an admissible control: $|u|_{L^\infty} \leq M$.

Lemma (A. P., C. Pignotti (2021))

Let $\{x_i\}_{i=0,1,\dots,N}$ *be the solution to* (3.27)-(3.28)*, with control as in* (3.29)*.*
Let $x_{0,0}, x_{i,0} : [-\bar{\tau}, 0] \to \mathbb{R}^d$ *be continuous initial data and let*

$$R := \max_{s\in[-\bar{\tau},0]} \max_{i=0,1,\dots,N} |x_i(s)|.$$

Then,

$$\max_{i\in\{0,1,\dots,N\}} |x_i(t)| \leq R, \tag{3.30}$$

for any $i \in \{1, \dots, N\}$ *and for any* $t \geq 0$*.*

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- The constant R in the previous lemma can be replaced, as before, by the maximal relative distance among the initial opinions.
- This lemma gives us a lower bound of ϕ, namely

$$\phi(|x_0(t - \tau(t)) - x_i(t)|) \geq \phi(2R) > 0,$$

for any $t \geq 0$ and for any $i \in \{1, ..., N\}$.

- Since d_0 may be not differentiable, we will use the upper Dini derivative defined as

$$D^+F(t) := \limsup_{h \to 0+} \frac{F(t+h) - F(t)}{h},$$

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Lemma (A. P., C. Pignotti (2021))

Let $\{x_i\}_{i=0,1,\dots,N}$ *be the solution to* (3.27)-(3.28)*, with control as in* (3.29). *Define*

$$\sigma_\tau(t) := \int_{t-\bar\tau}^{t} \left[\max_{j\in\{1,\dots,N\}} \left| \frac{dx_j}{ds}(s) \right| + |u(s)| \right] ds, \quad \forall\, t \geq \bar\tau.$$

Then, for any $t \geq \bar\tau$,

$$D^+d_0(t) \leq -\frac{\gamma}{2}\phi(2R)d_0(t) + R_\gamma \sigma_\tau(t), \tag{3.31}$$

where $R_\gamma := \gamma LR + \gamma + 1$.

Lemma (A. P., C. Pignotti (2021))

Let $\{x_i\}_{i\in\{0,1,\dots,N\}}$ *be the solution to* (3.27)-(3.28)*, with control as in* (3.29)*. Then, for any* $t \geq \bar{\tau}$*, we have*

$$\left|\frac{dx_0}{dt}(t)\right| \leq \gamma d_0(t) + \gamma\sigma_\tau(t) \tag{3.32}$$

and

$$\max_{i\in\{1,\dots,N\}}\left|\frac{dx_i}{dt}(t)\right| \leq (2+\gamma)d_0(t) + (1+\gamma)\sigma_\tau(t). \tag{3.33}$$

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Theorem (A. P., C. Pignotti (2021))

Let $\{x_i\}_{i\in\{0,1,\dots,N\}}$ *be the solution to* (3.27)-(3.28)*, with control as in* (3.29)*. If*

$$\bar{\tau} < \ln\left(1 + \frac{\gamma\phi(2R)}{\gamma(1+2\gamma)\phi(2R) + 4(1+\gamma)R_\gamma}\right), \tag{3.34}$$

then there exist $\beta^*, C > 0$ *such that, for any* $t \geq 0$*,*

$$d_0(t) \leq Ce^{-\beta^* t}. \tag{3.35}$$

Proof.

- Define

$$L(t) := d_0(t) + \beta \int_{t-\bar{\tau}}^{t} e^{-(t-s)} \int_{s}^{t} \left(\max_{j\in\{1,\dots,N\}} \left| \frac{dx_j}{d\sigma}(\sigma) \right| + \left| \frac{dx_0}{d\sigma}(\sigma) \right| \right) d\sigma ds, \quad t \geq \bar{\tau}.$$

- By differentiating in time, we can find a constant $\beta^* > 0$ such that

$$D^+ L(t) \leq -\beta^* L(t).$$

- Then, we can immediately conclude that

$$d_0(t) \leq L(t) \leq Ce^{-\beta^* t}, \quad \forall\, t \geq \bar{\tau}.$$

Local controllability

- Now, the aim is to show that, under some assumptions on the parameters and the communication rates, there exists an admissible control $u : [0, +\infty) \to \mathbb{R}^d$ which steers all agents to a given state $x^* \in \mathbb{R}^d$ (leader's target), namely

$$\lim_{t \to +\infty} x_i(t) = x^*, \quad \forall\, i \in \{0, 1, ..., N\}.$$

Lemma

Let $\{x_i\}_{i \in \{0,1,\dots,N\}}$ *be the solution to* (3.27)-(3.28)*, with* $u(t) = 0$ *for any* $t \geq 0$ *and* $x_{0,0}(s) = x_0$*, for any* $s \in [-\bar{\tau}, 0]$*. If for any* $i \in \{1, ..., N\}$

$$\max_{s \in [-\bar{\tau}, 0]} |x_i(s) - x_0| \leq \frac{\delta}{2},$$

and the following assumption on the parameters

$$\phi\left(\frac{\delta}{2}\right) > \frac{1}{2\gamma}, \tag{3.36}$$

is satisfied, then there exist two constants $C, K > 0$ *such that*

$$|x_i(t) - x_0| \leq Ce^{-Kt}, \tag{3.37}$$

for any $i \in \{1, ..., N\}$ *and for any* $t \geq 0$.

The key of the proof is the Halanay lemma for delay differential inequalities.

Lemma

Let $x(\cdot)$ *be a nonnegative continuous function satisfying*

$$\frac{dx}{dt}(t) \leq -a(t)x(t) + b(t)\left(\sup_{t-\tau(t)\leq s\leq t} x(s)\right), \quad t > t_0$$

where $a(t),\ b(t)$ *are nonegative, continuous and bounded functions. Suppose*

$$a(t) - b(t) \geq \sigma > 0, \quad t \geq 0.$$

Then, there exists $\mu > 0$ *such that*

$$x(t) \leq \left(\sup_{t_0-\bar{\tau}\leq s\leq t_0} x(s)\right) e^{-\mu(t-t_0)}, \quad t > t_0.$$

Lemma

Let $\{x_i\}_{i\in\{0,1,\dots,N\}}$ *be the solution to* (3.27)-(3.28). *Let* $\xi \in \mathbb{R}^d$. *Then, there exists a control* $u(\cdot) : [0,+\infty) \to \mathbb{R}^d$ *such that* x_0 *reaches* ξ *in finite time, namely there exists* $t_0 > 0$ *such that* $x_0(t_0) = \xi$. *Furthermore, if*

$$\tilde{R} := \max_{i\in\{0,1,\dots,N\}} \max_{s\in[-\bar{\tau},0]} |x_i(s) - \xi|,$$

then,

$$\max_{i\in\{0,1,\dots,N\}} |x_i(t) - \xi| \leq \tilde{R}, \tag{3.38}$$

for any $t \geq 0$.

The control u is of the form

$$u(t) := \begin{cases} M\frac{\xi - x_0(t)}{|\xi - x_0(t)|}, \text{ if } x_0(t) \neq \xi, \\ 0, \text{ if } x_0(t) = \xi, \end{cases} \tag{3.39}$$

for any $t \geq 0$.

Theorem (A. P., C. Pignotti (2021))

For any $\bar{x} \in \mathbb{R}^d$*, if the maximal time delay* $\bar{\tau}$ *satisfies* (3.34) *and* (3.36) *holds, then there exists an admissible control* u *which steers all agents to* $\bar{x}$*, namely*

$$\lim_{t \to +\infty} x_i(t) = \bar{x}, \tag{3.40}$$

for any $i \in \{0, 1, ..., N\}$*.*

Proof.

The proof is based on an iterative argument in which we use previous lemmas until we reach the prefixed state $\bar{x}$.

Numerical simulations

- Take $d = 1$ and $N = 50$ agents with constant initial data given by

$$x_i(t) = (-1)^i \frac{i}{50},$$

for any $i \in \{1, ..., 50\}$ and $t \leq 0$.

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- Choose $\delta = 1$, $r = 2$ and a_{ij} of the form

$$a_{ij}(t) = \begin{cases} 1, & \text{if } |x_j(t-\tau) - x_i(t)| \leq 1, \\ 2 - |x_j(t-\tau) - x_i(t)| & \text{if } |x_j(t-\tau) - x_i(t)| \in (1, 2), \\ 0 & \text{if } |x_j(t-\tau) - x_i(t)| > 2, \end{cases}$$

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for any $t \geq 0$ and $i, j \in \{1, ..., 50\}$.

- Take $\gamma = 1$ and $\phi(\cdot)$ as the classical influence function

$$\phi(s) = \frac{1}{(1+s^2)^{3/2}},$$

for any $s \geq 0$.

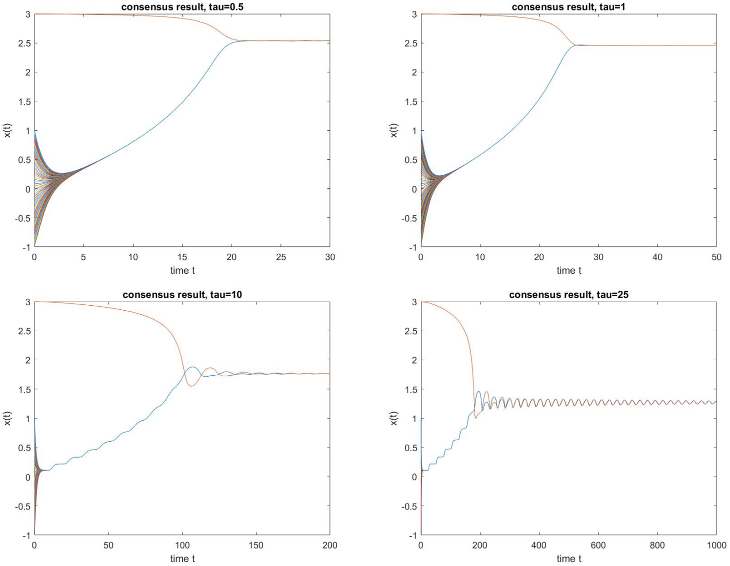

Figure: Time evolution of solutions with different values of time delays; $\tau = 0.5$ (top left), $\tau = 1$ (top right), $\tau = 10$ (bottom left), $\tau = 25$ (bottom right), corresponding to control function u as in (3.29).

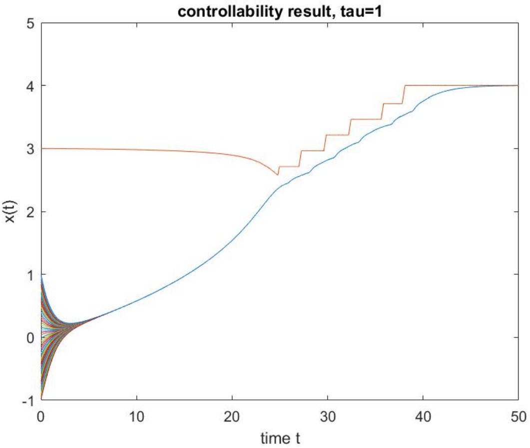

Figure: Local controllability result applied to system (3.27) with $\tau = 1$.

Thank you for your attention!