School-Workshop on
Analysis, Control and Inverse Problems for Diffusive Systems
with Application to Natural and Social Sciences

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The Maximum Principle for Lumped-Distributed Control Systems

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Outline

* Examplar Problem

* Infinite Dimensional Optimal Control

* Lumped-Distributed Optimal Control Problems

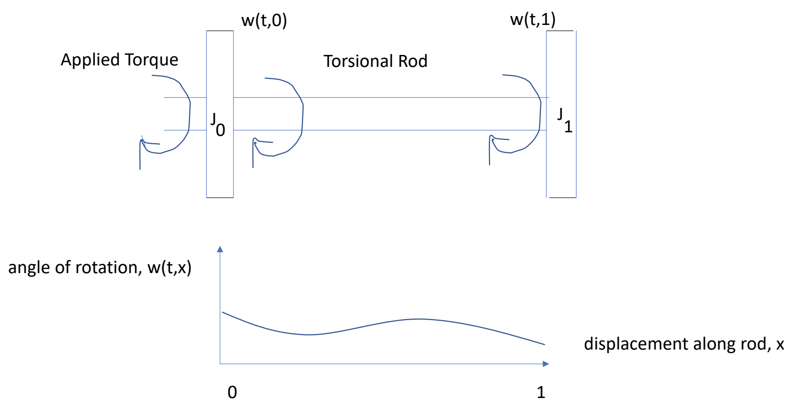
* A Maximum Principle

Lumped-Distributed Control Systems

Assemblages of interconnected subsystems with finite and infinite dimensional state spaces.

Such a system arises e.g. in robotics (control of masses connected by flexible rods).

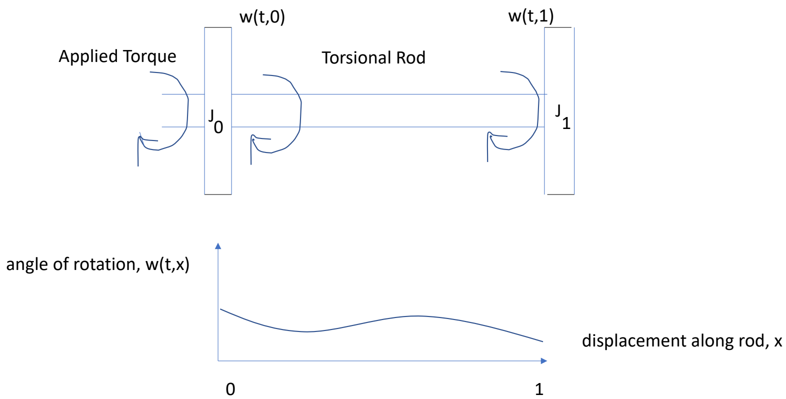
Example



Mechanical system with flexible connectors.

Lumped-Distributed Control Systems

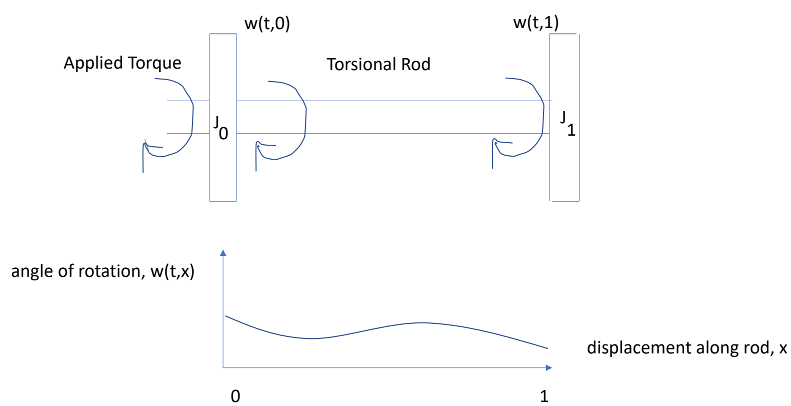
A left and a right inertial mass connected by a rod with rotational flexibility and uniform circular cross-section:



the lumped components:
the two inertial masses (the states are the angular displacements and velocities)
the distributed component:
the rod (the state is the angular strain along the rod and its time rate of change)
the control:
exogenous torque applied to the left inertial mass

Lumped-Distributed Control Systems

A left and a right inertial mass connected by a rod with rotational flexibility and uniform circular cross-section:



the lumped components:
the two inertial masses (the states are the angular displacements and velocities)

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the rod (the state is the angular strain along the rod and its time rate of change)

the control:
exogenous torque applied to the left inertial mass

State variables:
$v(y)$: angular velocity at point y along the rod $(0 \leq y \leq 1)$
$s(y)$: torsional strain at point y along the rod $(0 \leq y \leq 1)$
θ: angular velocity of left inertial mass
ψ: angular velocity of right inertial mass

Lumped-Distributed Control Systems

State $x = (v, s, \phi, \psi)$ evolves in the Hilbert space

$$X = L^2(0,1) \times L^2(0,1) \times \mathbb{R} \times \mathbb{R}$$

as mild solution of

$$\dot{x}(t) = \mathcal{A}x(t) + M \circ f(\Lambda \circ x(t), u(t)) \,,$$

where

$$\mathcal{A}\begin{bmatrix} v \\ s \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} c^2 s_y(y) \\ v_y(s) \\ k_f s(0) \\ -k_1 s(1) \end{bmatrix}$$

$c^2 := G/\rho,\ k_f := J_0^{-1}, k_1 := J_1^{-1}$

J_0, J_1: moments of inertia of the masses about the axis of rotation;

G: modulus of rigidity

ρ: mass per unit length

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{(v, s, \theta, \psi) \in W^{1,2}(0,1) \times W^{1,2}(0,1) \times \mathbb{R} \times \mathbb{R} \,:\, v(0) = \theta \text{ and } v(1) = \psi\right\}$$

- ⋆ $\mathcal{A}$ is dissipative
- ⋆ $\mathcal{R}(\mathcal{A} - \lambda_0 I) = X$

$\Longrightarrow$ $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup

from the Lumer-Phillips Theorem

Lumped-Distributed Control Systems

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$$\dot{x}(t) = \mathcal{A}x(t) + M \circ f(\Lambda \circ x(t), u(t)),$$

where

$$\mathcal{A} \begin{bmatrix} v \\ s \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} c^2 s_y(y) \\ v_y(s) \\ k_f s(0) \\ -k_1 s(1) \end{bmatrix}$$

$c^2 := G/\rho$, $k_f := J_0^{-1}$, $k_1 := J_1^{-1}$

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G: modulus of rigidity

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Inhomogeneous term $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$:

$$f(z_3, z_4, u) = (f_1(u), f_2(z_4)) := (u, -(z_4)^3) \qquad \text{non-linear damping}$$

Linear mapping $M : \mathbb{R} \times \mathbb{R} \to L^2 \times L^2 \times \mathbb{R} \times \mathbb{R}$:

$$M(f_1, f_2) := (0, 0, f_1, f_2) \qquad \text{immersion}$$

Linear mapping $\Lambda : L^2 \times L^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$:

$$\Lambda(z_1, \ldots, z_4) := (z_3, z_4) \qquad \text{projection}$$

Infinite Dimensional Optimal Control Problems

Minimize

$$g(x(T))$$

among solutions of

$$\dot{x}(t) = \mathcal{A}x(t) + f(t, x(t), u(t))$$

distinguishing feature:
state space is an infinite
dimensional vector
space

Infinite Dimensional Optimal Control Problems

Minimize

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$$\dot{x}(t) = \mathcal{A}x(t) + f(t, x(t), u(t))$$

distinguishing feature: state space is an infinite dimensional vector space $\rightarrow$ dynamics governed by PDEs: hyperbolic, parabolic... with boundary and distributed control

Infinite Dimensional Optimal Control Problems

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distinguishing feature: state space is an infinite dimensional vector space	$\rightarrow$	dynamics governed by PDEs: hyperbolic, parabolic... with boundary and distributed control	$\rightarrow$	applications to models describing many physical phenomena: control of thermal, electro-magnetic, flow and elastic deformation systems... control of nuclear fusion reactors, bio-reactors, wind/wave energy devices...

Infinite Dimensional Optimal Control Problems

Minimize

$$g(x(T))$$

among solutions of

$$\dot{x}(t) = \mathcal{A}x(t) + f(t, x(t), u(t))$$

- $\star$ $\mathcal{A}$ generates a strongly continuous semigroup $S(t)$ on X;
- $\star$ X is an infinite dimensional separable Hilbert space;
- $\star$ $f : I \times X \times Z \to X$;
- $\star$ $g : X \to \mathbb{R}$.

Infinite Dimensional Optimal Control Problems

Minimize

$$g(x(T))$$

among solutions of

$$\dot{x}(t) = \mathcal{A}x(t) + f(t, x(t), u(t))$$

satisfying $x(0) = x_0, \qquad x(T) \in C, \qquad x(t) \in K, \quad t \in I = [0, T]$

$x \in \mathcal{C}(I, X)$ is a mild solution (for given $u(\cdot)$) if

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x(s), u(s))ds$$

Fattorini, Infinite-dimensional optimization and control theory (1999)

(Butkovskii; Lions; Curtain, Pritchard; Cannarsa, Frankowska; Bensoussan, Da Prato, Delfour, Mitter; Barbu; Li, Yong; Lasiecka, Triggiani; Tröltzsch; ...)

Lumped-Distributed Optimal Control Problems

Use abstract infinite dimensional approach for lumped-distributed systems in

$$X = \mathbb{R}^n \times \mathcal{X}$$

$\mathbb{R}^n$ is lumped state space, $\mathcal{X}$ (Hilbert) is distributed state space.

Easier to regard $\mathbb{R}^n$ as a finite dimensional projection of X.

Model (as in earlier example):

$$\dot{x}(t) = \mathcal{A}x(t) + M \circ f(\Lambda \circ x(t), u(t))$$

the nonlinear term depends on lumped variables

Λ and M, a finite dimensional projection and an immersion

Lumped-Distributed Optimal Control Problems

- ⋆ lumped-distributed controlled evolution equation model
- ⋆ end-point constraint / cost depend on lumped state variables

$$
(P)\begin{cases}
\text{Minimize } g(\Lambda \circ x(T)) \\
\text{over functions } x \in \mathcal{C}(I;X) \text{ and measurable functions } u : I \to \mathbb{R}^m \text{ s.t.} \\
\dot{x}(t) = \mathcal{A}x(t) + M \circ f(t, \Lambda \circ x(t), u(t)), \quad \text{on } I, \\
u(t) \in U, \quad \text{for a.e. } t \in I, \\
\Lambda \circ x(T) \in C, \\
x(0) = x_0,
\end{cases}
$$

New data ingredients: closed set $C \subset \mathbb{R}^n$ (end-point constraint).
We can consider a more general case: $h(\Lambda \circ x(t)) \leq 0$ (pathwise state-constraint)

Lumped-Distributed Optimal Control Problems

Applications:

mechanical energy transmission systems (as wind generators);

automobile and aeronautical engineering;

communication systems (a transmission line has an active load);

thermal systems (a distributed thermal channel interacts with heat sinks and sources);

optimal control of hereditary systems (when the dynamic constraint is reformulated as a ‘delay-free’ evolution equation).

References:

Zabczyk (On Decomposition of Generators (1978)): examples where boundary controls to a distributed system are applied as the output of a lumped system.

Ozbay, Smith, Tannenbaum (Mixed sensitivity optimization for a class of unstable infinite dimensional systems (1993)); Curtain, Zhou (A weighted mixed-sensitivity H-infinity control design for irrational transfer matrices (1996): for linear systems, a matrix transfer function, relating input to output, whose entries are irrational functions.

In our problem the cost is expressed in terms of the projection of the infinite dimensional state onto a finite dimensional subspace: we can interpret this projection as the output of the system.

A maximum Principle

Take a minimizer $(\bar{x}, \bar{u})$. Then, $\exists\, \lambda \geq 0$, $p \in \mathcal{C}(I; \mathbb{R}^\kappa)$ such that:

(a) (multiplier non-triviality)

$$(\lambda, p) \neq (0, 0)$$

(b) (costate equation)

$$p(t) = M^* S(T-t)^* \Lambda^* p(T) + \int_t^T M^* S^*(s-t) \Lambda^* \xi(s) ds$$

(c): (transversality condition)

$$-p^T(T) \in \lambda \partial g(\Lambda \circ \bar{x}(T)) + N_C(\Lambda \circ \bar{x}(T))$$

(d): (Weierstrass condition)

$$p(t) \cdot f(t, \Lambda \circ \bar{x}(t), \bar{u}(t)) = \sup_{u \in U} p(t) \cdot f(t, \Lambda \circ \bar{x}(t), u))$$

$$\xi^T(s) \in \text{co}\, \partial_z \Big(f(s, \bar{z} = \Lambda \circ \bar{x}(s), \bar{u}(s)) \cdot p(s) \Big)$$

A maximum Principle

Comments:

- ⋆ The necessary conditions of our work cannot be deduced from those derived for general infinite dimensional problems in earlier literature:

  less restrictive hypotheses concerning the control and endpoint constraint sets, state constraint functions and regularity of the data.

- ⋆ A number of difficulties (connected with the lack of compactness):

  perturbational methods, based on approximation and passage to the limit, may fail (a family of Lagrange multipliers constructed as the limit of non-trivial Lagrange multipliers for approximating problems, may be trivial);

  perturbational methods that allow for non-smoothness of the data may also fail.

  We use the 'finite dimensional' aspects of our problem formulation.

- ⋆ In our necessary conditions, the costate evolves in a finite dimensional space:

  it can be exploited to construct computationally efficient schemes for solving the optimal control problem.

Decomposition

Fix a measurable $u : I \to U$.

Then, there exists a unique $z \in \mathcal{C}(I; \mathbb{R}^n)$ of

$$z(t) = \Lambda \circ S(t)x_0 + \int_0^t \Lambda \circ S(t-s)M \circ f(s, z(s), u(s))ds \qquad \text{for all } t \in I.$$

Furthermore

$$z(t) = \Lambda \circ x(t) \qquad \text{for all } t \in I$$

where x is the unique solution to the controlled evolution equation

$$x(t) = S(t)x_0 + \int_0^t S(t-s)M \circ f(s, \Lambda \circ x(s), u(s))ds \qquad \text{for all } t \in I.$$

Reformulation

The cost / constraint involve projection $z = \Lambda(x)$ of state.

So we can reformulate the control problem in terms of z:

$$(E)\begin{cases} \text{Minimize } g(z(T)) \\ \text{over functions } z \in \mathcal{C}(I;\mathbb{R}^n) \text{ and measurable functions } u : I \to \mathbb{R}^m \text{ s.t.} \\ z(t) = \Lambda \circ S(t)x_0 + \int_0^t \Lambda \circ S(t-s)M \circ f(s, z(s), u(s))ds, \quad \text{for all } t \in I\,, \\ u(t) \in U\,, \quad \text{for a.e. } t \in I\,, \\ z(T) \in C\,, \\ z(0) = \Lambda(x_0)\,. \end{cases}$$

- ⋆ $(\bar{x}, \bar{u})$ is minimizer for (P) $\Longrightarrow$ $(\bar{z} := \Lambda \circ \bar{x}, \bar{u})$ is minimizer for (E)
- ⋆ derive necessary condition for (E) and interpret in terms of data foe (P)

Proof

A generalization of Clarke's nonsmooth maximum principle (The maximum principle under minimal hypotheses (1976)) to optimal control problems, in which the dynamic constraint takes the form of a semilinear evolution equation.

Assumptions:

- ⋆ there exists $k_f \in L^1(I;\mathbb{R}^+)$ such that

$$|f(t,z,u) - f(t,z',u)| \leq k_f(t)|z - z'|, \text{ for all } z, z' \in \bar{z}(t) + \epsilon B\,, \text{ a.e. } t \in I$$

- ⋆ there exists $c_f \geq 0$ such that

$$|f(t,z,u)| \leq c_f, \text{ for all } z \in \bar{z}(t) + \epsilon B \text{ and } u \in U, \text{ a.e. } t \in I$$

- ⋆ g is Lipschitz continuous on $(\bar{z}(0), \bar{z}(T)) + \epsilon(B \times B)$

Proof

A generalization of Clarke's nonsmooth maximum principle (The maximum principle under minimal hypotheses (1976)) to optimal control problems, in which the dynamic constraint takes the form of a semilinear evolution equation.

Notation:

- proximal normal cone:

$$N_C^P(x) := \Big\{\zeta \in \mathbb{R}^n : \exists\, \epsilon > 0 \text{ and } M > 0 \text{ s.t. } \zeta \cdot (y - x) \le M|x - y|^2,\ \forall\, y \in C \cap x + \epsilon B\Big\}$$

- limiting normal cone:

$$N_C(x) := \Big\{ \lim_{i\to\infty} \zeta_i : \zeta_i \in N_C^P(x_i),\ x_i \in C,\ x_i \to x\Big\}$$

- proximal subdifferential:

$$\partial^P f(x) := \Big\{\zeta \in \mathbb{R}^n : \exists\, \sigma > 0 \text{ and } \epsilon > 0 \text{ s.t. } f(y) - f(x) \ge \zeta \cdot (y - x) - \sigma|y - x|^2,\ \forall\, y \in x + \epsilon B\Big\}$$

- limiting subdifferential:

$$\partial f(x) := \Big\{ \lim_{i\to\infty} \zeta_i : \zeta_i \in \partial^P f(x_i), x_i \to x, f(x_i) \to f(x)\Big\}$$

Proof: perturbational method allowing for non-smooth data

Quadratic inf-convolution techniques: for given $k \in \mathbb{N}$, define

$$g_k(z) := \inf_{y \in \mathbb{R}^n} \{g(y) + k|y - z|^2\}$$

Properties:

- ⋆ g_k is Lipschitz continuous with Lipschitz constant k_g (for k_g as in (H4))
- ⋆ $g(z) \leq g_k(z) + \frac{k_g}{k}$
- ⋆ $g_k(z') - g_k(z) \leq \eta_k \cdot (z' - z) + k|z' - z|^2$, for each $z' \in \mathbb{R}^n$
- ⋆ $\eta_k \in \partial_P g(y)$
- ⋆ $|y - z| \leq \frac{k_g}{k}$

$(\eta_k := -2k(y - z)$

Proof: perturbational method allowing for non-smooth data

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$$g_k(z) := \inf_{y \in \mathbb{R}^n} \left\{ g(y) + k|y - z|^2 \right\}$$

$$(E_k) \begin{cases} \text{Minimize } \; g_k(z(T)) \underbrace{+k \displaystyle\int_0^T k_f(t)|z(t) - y(t)|^2 dt}_{\text{penalty term}} \\ \text{over functions } (z, y, u) \text{ such that } y \text{ and } u \text{ are measurable functions and} \\ z(t) = \Lambda \circ S(t)x_0 + \int_0^t \Lambda \circ S(t-s) M \circ f(s, y(s), u(s)) ds, \quad \text{for any } t \in I \\ u(t) \in U, y(t) \in \mathbb{R}^n, \quad \text{a.e. } t \in I \\ \int_I k_{\tilde{f}}(t)|y(t)|dt < \infty \end{cases}$$

Proof: perturbational method allowing for non-smooth data

Ekeland's Theorem:

- ⋆ $(\bar{z}, \bar{z}, \bar{u})$ is a ϵ_k-minimizer for problem (E_k)
- ⋆ there exists a minimizer (z_k, y_k, u_k) for the problem: minimize

$$g_k(z(T)) + k\int_0^T k_f(t)|z(t) - y(t)|^2 dt$$
$$+\sqrt{\epsilon_k}\Big(\int_I k_f(t)|y(t) - y_k(t)|dt + \text{meas}\{t \in I : u(t) \neq u_k(t)\}\Big)$$

Proof: perturbational method allowing for non-smooth data

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$$+\sqrt{\epsilon_k}\Big(\int_I k_f(t)|y(t) - y_k(t)|dt + \text{meas}\{t \in I : u(t) \neq u_k(t)\}\Big)$$

- ⋆ by the quadratic inf-convolution properties (z_k, y_k, u_k) minimizes

$$\zeta_k \cdot (z(T) - z_k(T)) + k|z(T) - z_k(T)|^2 + k\int_I k_f(t)|y(t) - z(t)|^2 dt$$
$$+\sqrt{\epsilon_k}\Big(\int_I k_f(t)|y(t) - y_k(t)|dt + \text{meas}\{t \in I : u(t) \neq u_k(t)\}\Big)$$

$\zeta_k \in \partial_P g(y_k^1)$, for some $y_k^{(1)} \in z_k(T) + \frac{k_g}{k}\mathcal{B}$.

Proof: perturbational method allowing for non-smooth data

By using needle variations:

$$p_k(t) = -M^* S^*(T-t)\Lambda^* \zeta_k + 2k \int_t^T M^* S^*(s-t)\Lambda^* \Big(k_f(s)(y_k(s) - z_k(s))\Big) ds$$

satisfies

$$p_k(t) \cdot f(t, y_k(t), u_k(t)) \geq p_k(t) \cdot f(t, y_k(t), u) - \sqrt{\epsilon_k}\,, \text{ for all } u \in U, \text{ a.e. } t \in I\,,$$

and

$$-2k k_f(t)(y_k(t) - z_k(t))^T \in \partial_z \Big(- p_k(t) \cdot \tilde{f}(t, y_k(t), u_k(t))\Big) + \sqrt{\varepsilon_k} B$$

Proof: perturbational method allowing for non-smooth data

By using needle variations:

$$p_k(t) = -M^*S^*(T-t)\Lambda^*\zeta_k + 2k\int_t^T M^*S^*(s-t)\Lambda^*\Big(k_f(s)(y_k(s)-z_k(s)\Big)ds$$

satisfies

$$p_k(t)\cdot f(t,y_k(t),u_k(t)) \geq p_k(t)\cdot f(t,y_k(t),u) - \sqrt{\epsilon_k}\,, \quad \text{for all } u\in U, \ \text{ a.e. } t\in I\,,$$

and

$$-2kk_f(t)(y_k(t)-z_k(t))^T \in \partial_z\Big(-p_k(t)\cdot\tilde{f}(t,y_k(t),u_k(t))\Big) + \sqrt{\varepsilon_k}B$$

passing to the limit, as $k\to\infty$, we get

$$-p(t) = M^*S(T-t)^*\Lambda^*\zeta - \int_t^T M^*S(s-t)^*\Lambda^*\alpha(s)ds, \qquad \text{for each } t\in I,$$

for some vector $\zeta\in\mathbb{R}^n$ and some measurable function α such that

$$\zeta^T\in\partial_z g(\bar{z}(T)) \qquad \text{and} \qquad \alpha^T(t)\in \text{co}\,\partial_z p(t)\cdot f(t,\bar{z}(t),\bar{u}(t)) \text{ a.e.},$$

and

$$p(t)\cdot f(t,\bar{z}(t),\bar{u}(t)) \geq p(t)\cdot f(t,\bar{z}(t),u)), \qquad \text{for all } u\in U, \ \text{ a.e. } t\in I.$$

Proof: perturbational method allowing end-point contsraint

Ekeland's Theorem:

⋆ $(\bar{z}, \bar{u})$ is a ϵ_k-minimizer for problem: minimize

$$\max\left\{g(z) - g(\bar{z}(T)) + \epsilon_k, d_C(z)\right\}$$

⋆ there exists a minimizer (z_k, u_k) for the problem: minimize

$$\max\left\{g(z) - g(\bar{z}(T)) + \epsilon_k, d_C(z)\right\} + \text{meas}\left\{t : u_k(t) \neq \bar{u}(t)\right\}$$

Proof: perturbational method allowing end-point contsraint

for some $\alpha_k^T(t) \in \text{co}\partial_z p_k(t) \cdot f(t, z_k(t), u_k(t))$, $\zeta_k^T \in \partial \max \left\{ g(z_k(T)) - g(\bar{z}(T)) + \gamma_k, d_C(z_k(T)) \right\}$

$$-p_k(t) = M^* S^*(T-t)\Lambda^* \zeta_k - \int_t^T M^* S^*(s-t)\Lambda^* \alpha_k(s) ds$$

satisfies

$$p_k(t) \cdot f(t, z_k(t), u_k(t)) \geq p_k(t) \cdot f(t, z_k(t), u) - \sqrt{\varepsilon}_k, \quad \text{for all } u \in U, \text{ a.e. } t \in I.$$

By optimality:

$$\max\{g(z_k(T)) - g(\bar{z}(T)) + \gamma_k, d_C(z_k(T))\} > 0 \quad \Longrightarrow \quad \zeta_k \in \lambda_k \partial g(z_k(T)) + (1-\lambda_k)\sigma_k$$

for some $\lambda_k \in [0,1]$ and $\sigma_k \in \partial d_C(z_k(T))$ such that $|\sigma_k| = 1$.

Passing to the limit: $\exists (p, \lambda)$ satisfying

(b) (costate equation)

$$p(t) = M^* S(T-t)^* \Lambda^* p(T) + \int_t^T M^* S^*(s-t)\Lambda^* \xi(s) ds$$

(d): (Weierstrass condition)

$$p(t) \cdot f(t, \bar{z}(t), \bar{u}(t)) = \sup_{u \in U} p(t) \cdot f(t, \bar{z}(t), u))$$

Proof: perturbational method allowing end-point contsraint

for some $\alpha_k^T(t) \in \text{co}\partial_z p_k(t)\cdot f(t,z_k(t),u_k(t))$, $\zeta_k^T \in \partial \max\left\{g(z_k(T)) - g(\bar{z}(T)) + \gamma_k, d_C(z_k(T))\right\}$

$$-p_k(t) = M^*S^*(T-t)\Lambda^*\zeta_k - \int_t^T M^*S^*(s-t)\Lambda^*\alpha_k(s)ds$$

satisfies

$$p_k(t)\cdot f(t,z_k(t),u_k(t)) \geq p_k(t)\cdot f(t,z_k(t),u) - \sqrt{\varepsilon}_k, \quad \text{for all } u\in U, \text{ a.e. } t\in I.$$

By optimality:

$$\max\{g(z_k(T)) - g(\bar{z}(T)) + \gamma_k, d_C(z_k(T))\} > 0 \quad \Longrightarrow \quad \zeta_k \in \lambda_k\partial g(z_k(T)) + (1-\lambda_k)\sigma_k$$

for some $\lambda_k \in [0,1]$ and $\sigma_k \in \partial d_C(z_k(T))$ such that $|\sigma_k| = 1$.

Passing to the limit:

$$(-p(T) =)\ \zeta^* \in \lambda\partial g(\bar{z}(T)) + (1-\lambda)\sigma,$$

$$\sigma \in \partial d_C(\bar{z}(T)) \subset N_C(\bar{z}(T)) \text{ such that } |\sigma| = 1$$

hence

(c): (transversality condition)

$$-p^T(T) \in \lambda\partial g(\Lambda\circ\bar{x}(T)) + N_C(\Lambda\circ\bar{x}(T))$$

Further, if $\lambda = 0$, then $|p(T)| = |\sigma| = 1$ and so $p \neq 0$, hence

(a) (multiplier non-triviality)

$$(\lambda, p) \neq (0,0)$$

Thank you for your kind attention