Stabilization from the boundary in a third order in time nonlinear dynamics with applications to nonlinear acoustics.

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Overview

- Motivation:Lithotripsy, High Frequency Focused Ultrasound.
- PDE models:Nonlinear Acoustics: Westervelt- Kuznetsov and Moore-Gibson-Thompson equations.

  - Westervelt/ Kuznetsov-2-nd order in time (infinite speed of propagation) -parabolic type
  - Moore-Gibson-Thompson equation -3-rd order in time (finite speed of propagation)-hyperbolic type

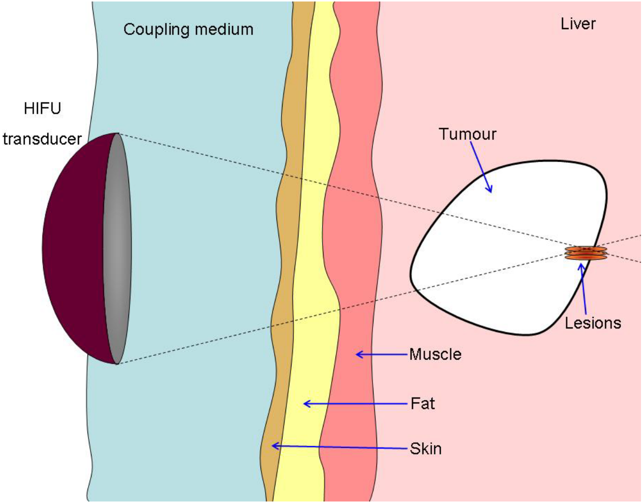
- Global well-posedness of these **quasilinear** PDE models
- Analysis when **the relaxation time** τ goes to zero. From propagation to diffusion.
- **Hidden regularity** from the boundary.
- Stabilization: **frictional, memory and boundary damping.**
- **Boundary control** [weak solutions] and feedback control.

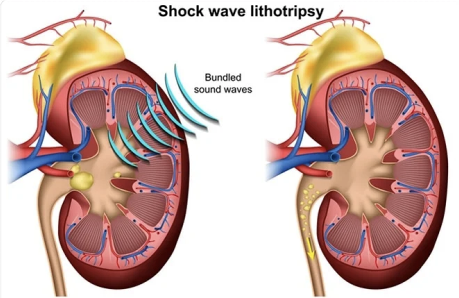
Challenges

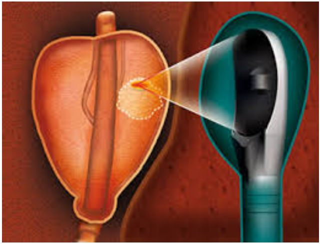
- **Nonlinear** wave propagation.
- **Singular perturbation** : Linear generator $\mathcal{A}_\tau$ becomes singular - from hyperbolic to parabolic.
- Weak solutions with **rough boundary data**.
- **Boundary Feedback Control Problems.** Nonstandard **Riccati Equations with Unbounded** coefficients.

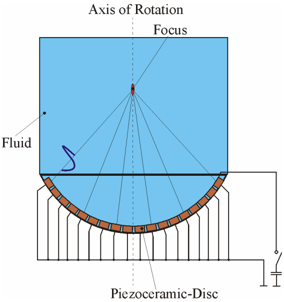
THE PLAN.

1. **PART I** - review of MGT dynamics- stability in critical case.
2. **PART II** - Optimal Boundary Control and Boundary Stabilizability.
3. **PART III** - Feedback boundary control and Nonstandard ARE.


Coupling medium
Liver
HIFU
transducer
Tumour
Lesions
Muscle
Fat
Skin


Shock wave lithotripsy
Bundled
sound waves





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optimal control of the excitation signal

Modeling

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\rho(\mathbf{v_t} + (\mathbf{v} \cdot \nabla)\mathbf{v}) = \nabla \cdot \mathbf{T}$$
$$\rho\theta(\eta_t + (\mathbf{v} \cdot \nabla)\eta) = -\nabla \cdot \mathbf{q} + \mathbf{T} : \mathbf{D} \tag{1}$$

η -entropy, **q**-heat flux, **D** -deformation tensor, **T**-Cauchy Poisson stress tensor. $p_\sim$ ... pressure fluctuation
v... acoustic particle velocity
ψ ...acoustic velocity potential
ρ ... mass density
c ... speed of sound
b ... diffusivity of sound
B/A ... parameter of nonlinearity
$\mathbf{v} = -\nabla\psi\,, \qquad \rho D_t \mathbf{v} = -\nabla p_\sim$
Lesser&Seebass 1968, Kuznetsov 1971

Fourrier's Law

$$\mathbf{q} = -\mathbf{K}\nabla\theta$$

Cattaneo Law

$$\tau \mathbf{q_t} + \mathbf{q} = -\mathbf{K}\nabla\theta$$

$\tau > 0$ small relaxation time parameter

Equations of Nonlinear Acoustics

Westervelt -Kuznetsov equation

$$D_t^2 p_\sim - c^2 \Delta p_\sim - bD_t \Delta p_\sim = \frac{1}{\rho c^2} D_t^2 \left((1 + \frac{B}{2A}) p_\sim^2 + |\rho c \mathbf{v}|^2 \right)$$

MGT-Moore -Gibson - Thompson equation

$$\tau D_t^3 p + D_t^2 p_\sim - c^2 \Delta p_\sim - bD_t \Delta p_\sim = \frac{1}{\rho c^2} D_t^2 \left((1 + \frac{B}{2A}) p_\sim^2 + |\rho c \mathbf{v}|^2 \right)$$

Modeling: Jordan Pedro, Ivan Christov, Christo Christov, Brian Straughan.

Westervalt Equation - Fourier's Law ,"infinite" speed of propagation

Westervelt equation with Dirichlet boundary conditions: $u(t,x)$ =acoustic pressure

$$\alpha D_t^2 u - c^2 \Delta u - bD_t \Delta u = kD_t^2(u^2) \textbf{ in } (0,T) \times \Omega$$

Rewrite as a **degenerate-quasilinear**

$$(\alpha - 2ku)D_t^2 u - c^2 \Delta u - bD_t \Delta u = 2k(D_t u)^2$$

$$\alpha(t,x)D_t^2 u - c^2 \Delta u - bD_t \Delta u = f(D_t u)$$

- Degenerate : $\alpha(t,x) = (\alpha - 2ku(t,x))$ can vanish
- Nonlinear term $f(D_t u) = 2k[(D_t u)^2 + uD_t^2 u]$
- $k = \frac{1}{c^2}(1 + \frac{B}{2A})$

M-T-G eq.- Cattaneo Law, "finite" speed of propagation

Let Ω be a bounded domain of $\mathbb{R}^n$ with a regular boundary Γ

$$\tau u_{ttt} + [\alpha - 2ku]u_{tt} - c^2\Delta u - b\Delta u_t = 2k(D_t u)^2 \quad (2)$$

$$u = 0 \ on \ \Gamma = \partial\Omega \quad (3)$$

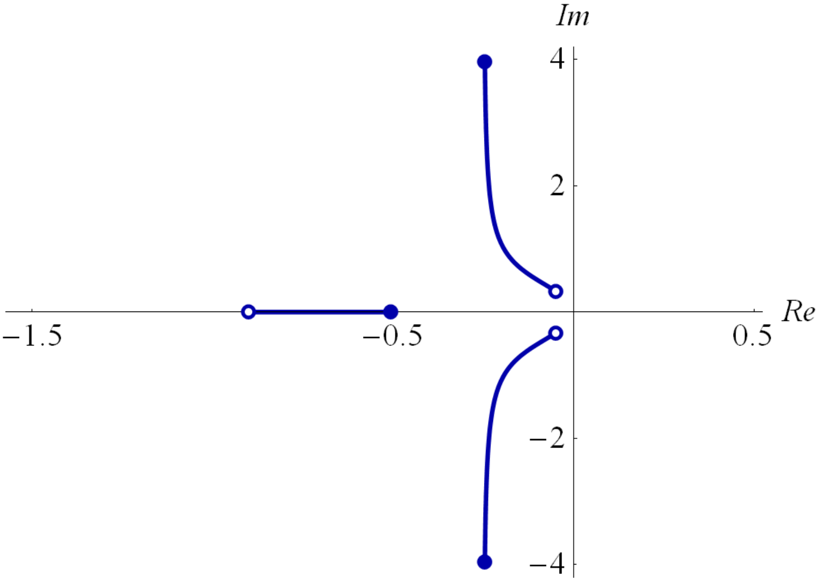
where in a physical context of the acoustic waves

- the variable u denotes a scalar acoustic velocity potential $\vec{v} = -\nabla u$ with $\vec{v}$ denoting the acoustic particle velocity.
- c^2 denotes the speed of sound ,
- τ denotes thermal relaxation resulting from **replacing Fourier's law by the Maxwell Cattaneo law**.
- The coefficient $b \equiv \delta + \tau c^2$ where δ is the diffusivity of the sound.
- The coefficient $\alpha > 0$ describes natural damping effects associated with an acoustic environment.

The presence of the **third time derivative is typical in Extended Irreversible Thermodynamics (EIT)** *a theory originally proposed to remove the unpleasant property of propagation of heat and velocity signals with an infinite velocity when Fourier-Navier-Stokes equations are used* **.** The guiding idea behind is that physical quantities such as thermodynamic fluxes typically given by constitutive relations, in EIT theory are governed by evolution equations with a suitable relaxation time τ.

- $\tau = 0$. This is **Parabolic like** Problem.
  Kuznetsov eq (Westervalt without the blue term) .
- $\tau > 0$. This is **Hyperbolic like** Problem.
  Moore-Thompson-Gibson equation.

:: **Case** $\tau > 0$ **first introduced by "Professor Stokes" in 1851** .

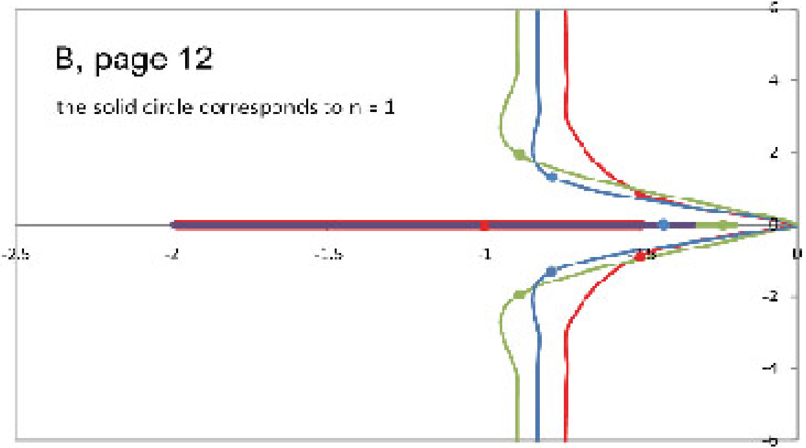

Im
4
2
−2
−4
Re
−1.5
−0.5
0.5

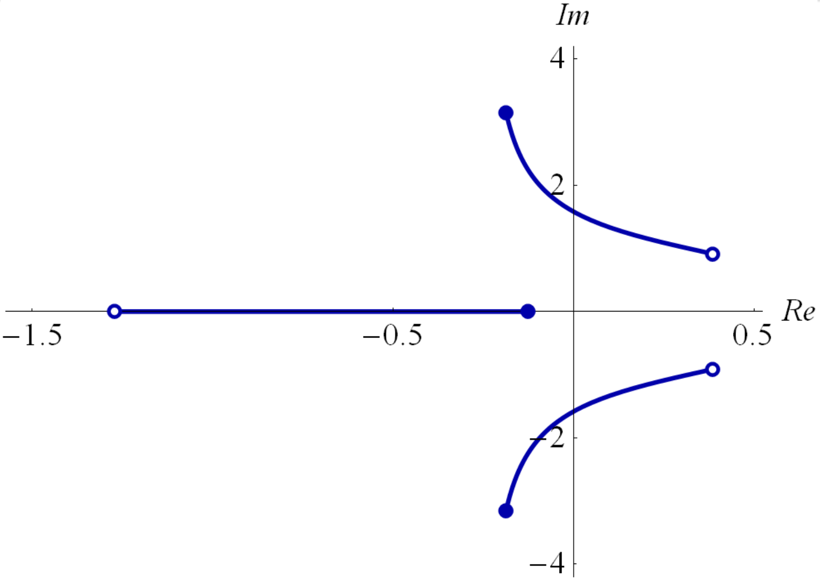
$\tau = c = 1$

$n = 0.032$ or $0.034, 1, 2, 3, \ldots 100$

$\alpha = 2$

$b = 2$	$b = 3$	$b = 5$
$\gamma = 1.5$	$\gamma = 1.66$	$\gamma = 0.8$





Parameter of stability $\gamma < 0$. $\gamma \equiv \alpha - \frac{\tau c^2}{b}$

Abstract formulations

$\tau = 0$- **Westervalt/ Kuznetsov**

$$(\alpha - 4ku)u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u_t = 4ku_t^2 \tag{4}$$

$$H \equiv D(\mathcal{A}^{1/2}) \times \mathcal{H}, \qquad H_1 \equiv D(\mathcal{A}) \times D(\mathcal{A}^{1/2})$$

$\tau > 0$- **MGT**

$$\tau u_{ttt} + (\alpha - 4ku)u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u_t = 4ku_t^2 \tag{5}$$

$$H \equiv D(\mathcal{A}^{1/2}) \times D(\mathcal{A}^{1/2}) \times \mathcal{H}, \qquad H_1 \equiv D(\mathcal{A}) \times D(\mathcal{A}^{1/2}) \times \mathcal{H}$$

$\mathcal{A}$ corresponds to negative Laplasjan with zero boundary conditions [Neuman/Dirichlet]

Theorem (THM 1: Linear stability)

Let $k = 0$.

1. $\tau = 0$. e^{At} *is exponentially stable iff* $\alpha > 0, b > 0$.
2. $\tau > 0$. e^{At} *is exponentially stable iff* $\gamma \equiv \alpha - \frac{\tau c^2}{b} > 0$.

Theorem (THM 2: Global solutions for nonlinear system.)

1. *When* $\tau = 0$, $\alpha > 0$ *there exists unique* **global** *solution provided the initial data are small with respect to* $D(\mathcal{A}) \times D(\mathcal{A}^{1/2})$ *or* $W_p^1 \times L_p$ *for* $p > max[n/2, n/4 + 1]$.
2. $\tau \geq 0$, $\gamma > 0$, *there exists unique* **global** *solution provided the initial data are* **small** *with respect to* $D(\mathcal{A}) \times D(\mathcal{A}^{1/2}) \times \mathcal{H}$.

McDevitt, Marchand, Triggiani, 2012 MMAS.
Kaltenbacher, IL, 2012, *MathNach*
Kaltenbacher, IL. M. Pospiesz , 2013 *MMMAS*
Meyer, Wilke, 2013, *AMO*

Parameter of stability: $\gamma \equiv \alpha - \frac{\tau c^2}{b}$

Energy functions

$$E_0(t) \equiv ||\mathcal{A}^{1/2} u(t)||^2 + ||\mathcal{A}^{1/2} u_t(t)||^2 + ||u_{tt}(t)||^2$$

$$E_1(t) \equiv ||\mathcal{A} u(t)||^2 + E_0(t)$$

$\gamma = \alpha - \frac{\tau c^2}{b}$, For $\gamma > 0$ and k =0, $E_i(t) \leq Ce^{-\omega t}$

Owing to **exponential stability with** $\gamma > 0$ of the linearization

Proof of global wellposedness of the **nonlinear problem** for **small** initial data is based on "barrier's method " used in hyperbolic quasilinear theory. Technical tools: a string of suitable estimates developed for the linearization. .

Stability for Critical JMGT

JMGT is not stable in the critical case $\gamma = \alpha - \tau \frac{c^2}{b} = 0$,

What if $\gamma = 0$

When $\gamma = 0$ then $E(t) \sim$ *const* for linear model.
Need to stabilize. Two options:

- **Memory** damping: $\int_0^t g(t-s)\mathcal{A}[au(s) + bu_t(s)]ds$.
  Filippo Del' Oro, Vittorino Pata, Xiaojun Wang, IL
- **Boundary** damping on a part of the boundary. with M. Bongarti and R. Triggiani. 2020 and M.Bongarti, I.L. J.Rodriquez 2021.

Diffusion versus propagation.

Asymptotic Analysis when $\tau \to 0$.

$$U \equiv (u, u_t, u_{tt}.)$$

Convergence of $U^\tau \to U^{\tau=0}$ **when** $\tau \to 0$**???? Where???**

- Wellposedness and regularity results for the JMGT (nonlinear) require some type of **smallness of the initial data** .
- **How small ???** So far:

$\longrightarrow$ wellposedness in $\mathbb{H}_1 \longleftrightarrow$ data small in $\mathbb{H}_1$

$\longrightarrow$ wellposedness in $\mathbb{H}_2 \longleftrightarrow$ data small in $\mathbb{H}_2$.

- In order to show convergence of the **nonlinear** semigroups in the phase space $\mathbb{H}_1$, one needs estimates in a **higher** topology: $\mathbb{H}_2$.
- Although $\mathbb{H}_2$ is dense in $\mathbb{H}_1$, if wellposedness in each space is **tied to smallness in that space**, one cannot use the density.

Why do we need to be careful with the density argument in the nonlinear environment? Because of strong limit process.

Weak versus Strong limit

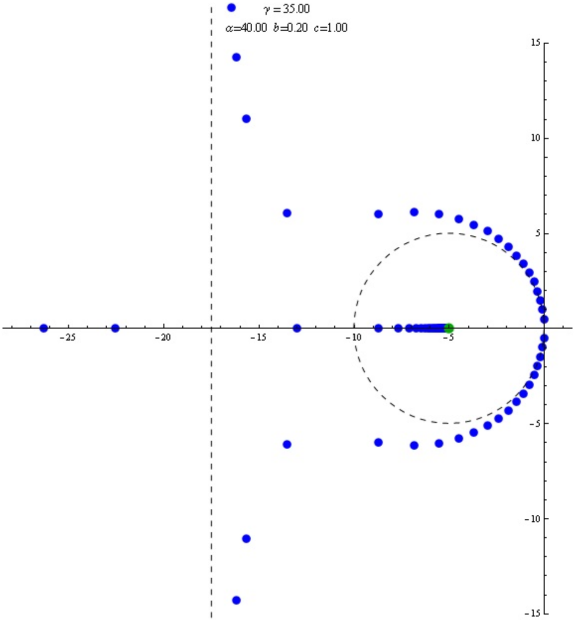
Singular generator A^τ:

$$A^\tau = \frac{-1}{\tau} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ c^2\mathcal{A} & \beta\mathcal{A} & \alpha I \end{pmatrix}$$

- B. Kaltenbacher and V. Nikolic: MMAS 2020, 2019)
  Weak convergence established.
- **Strong Convergence - Open Problem.**
- **Why OPEN?** Singular generator and need to handle quasilinearity at various topological levels.

To settle the problem three ingredients:

- (1) control of **singularity** in τ of u_{ttt},
- (2) **Tightness [reduction] of the "smallness"** to the base energy reflected in all uniform estimates of the energy. **How Small?**
- (3) **Invariance** of the "tightness" on the dynamics.



when $\tau \to 0$ the vertical line goes to $-\infty$.

Strong convergence

Theorem (Bongarti, Charoenphon, Lasiecka, JEE, 2020)

a) Rate of Convergence: Let $T > 0$ and let $U_0 \in \mathbb{H}_2 \sim H^2 \times H^2 \times L_2$ with $\|U_0\|_{\mathbb{H}_0^\tau} \leq \rho$ sufficiently small. Then there exists a τ-independent constant C_T such that

$$\left\|P(U^\tau(t, U_0)) - U^0(t, PU_0)\right\|^2_{H^2 \times H^1} \leqslant C_T \tau \|U_0\|_{H^2 \times H^2 \times H^1}$$

uniformly (in t) for $t \in [0, T]$.

b) Strong Convergence: Let $U_0 \in \mathbb{H}_1 = H^2 \times H^1 \times L_2$ with $\|U_0\|_{\mathbb{H}_0^\tau} \leq \rho$ for ρ as above. Then the following strong convergence

$$\left\|P(U^\tau(t, U_0)) - U^0(t, PU_0)\right\|_{H^2 \times H^1} \to 0 \quad \text{as} \quad \tau \to 0$$

holds uniformly on $[0, \infty)$.

Boundary Control Problem

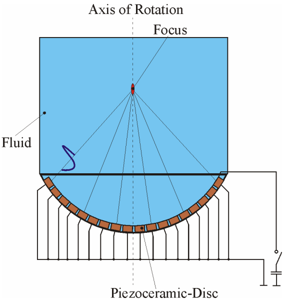
$$\tau D_t^3 u + D_t^2 u - c^2 \Delta u - b\Delta(D_t u) = k D_t^2 u^2 + \gamma_1 D_t^2 |\nabla(\int_0^t u \, d\tau)|^2 \text{ in } (0,T) \times$$

$$\frac{\partial u}{\partial n} + u = g \quad \text{on } (0,T) \times \Gamma_0 \quad \ldots \text{boundary excitatio}$$

$$D_t u + \frac{\partial u}{\partial n} = 0 \quad \text{on } (0,T) \times \Gamma_1 \quad \ldots \text{absorbing boundar}$$

$\gamma = \alpha - \frac{\tau}{c^2 b} \geq 0$**-including the critical case**

.



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optimal control of the excitation signal

Note that this is Neuman-Neuman configuration with a part which is nondissipated.

Optimal Boundary Control Problem

Past results: Finite horizon and smooth OPEN LOOP controls

$$\min_{g \in G^{ad}} J(g,u) \text{ s.t.}$$

$$\begin{aligned}
\tau D_t^3 u + D_t^2 u - c^2 \Delta u - b\Delta(D_t u) &= kD_t^2 u^2 + \gamma D_t^2 |\nabla(\int_0^t u\, d\tau)|^2 \text{ in } (0,T)\times \Omega \\
\frac{\partial u}{\partial n} &= g \qquad \text{on } (0,T)\times\Gamma_0 \quad \ldots \text{boundary excitation} \\
D_t u + c\frac{\partial u}{\partial n} &= 0 \qquad \text{on } (0,T)\times\hat{\Gamma}_1 \quad \ldots \text{absorbing boundary}
\end{aligned}$$

$$J(g,u) = \frac{1}{2}\int_0^T ||u - u_d||_{L_2}^2 + \frac{1}{2}\int_0^T ||g||_G^2$$

$$||g||_G := ||g||_{H^2(0,T;H^{-1/2}(\Gamma)} + ||g||_{H^1(0,T;H^{1/2}(\Gamma))})$$

Kaltenbacher, Clason, JMAA 2009, EECT 2015

Controls are required to have $3/2$ **derivatives on the boundary .**

GOAL : Non-smooth controls AND $T = \infty$, and $\tau \geq 0$, $\gamma \geq 0$.

$$\min_{g \in L_2(L_2)} J(g,u) \text{ s.t.}$$

$$\tau D_t^3 u + D_t^2 u - c^2 \Delta u - b\Delta(D_t u) = 0$$

$$\frac{\partial u}{\partial n} = g \qquad \text{on } (0,T) \times \Gamma_0 \quad \text{control}$$

$$D_t u + \frac{\partial u}{\partial n} = 0 \qquad \text{on } (0,T) \times \Gamma_1 \quad \text{absorbing BC}$$

$$J(g,u) = \frac{1}{2} \int_0^\infty ||u - u_d||^2_{L_2(\Omega)} + \frac{1}{2} \int_0^\infty |g|^2_{L_2(\Gamma_0)}$$

Two issues: (1) Stabilizability [$\gamma = 0$]; (2) Optimal Feedback Control.[$g = F(u, u_t, u_{tt})$] .

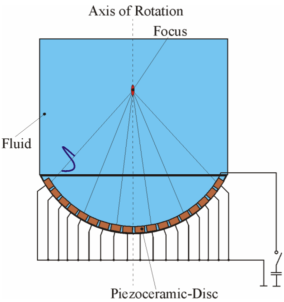
When $T < \infty$, $\tau > 0$-hyperbolic case. -F. Bucci , I.L *Optimization*, 2019. **Finite time horizon for a singular control problem.**

Theorem (F. Bucci, IL , Optimization, 2019)

- $y \equiv [u, u_t, u_{tt}]$. *Given* $y_0 \in [D(A^{2*})]'$ *and* $g(0) \in U$ *,there exists unique optimal control* $g^* \in L_2(U)$.
- $y^* \in C([0,T];[D(A^{2*})]')$, $g^* \in C([0,T];U)$.
  $Ry^* = u^* \in C(0,T;Y)$
- *The control* $g^* = F(y^*)$ *where* F *is given via appropriate [Differential Riccati Equation] with* $P(t)$ *solution to Diff. Riccati equation.*

In order to extend to **infinite time horizon**,

- Analysis of boundary dynamics.
- Analysis of Riccati equations with **unbounded** coefficients. Tool: DRE/ARE with "smoothing observation".
- Eliminate $g(0) \in U$. Modeling of boundary control with L_2 data.
- Needs uniform **boundary stability** of the linear dynamics. [Bongarti,



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optimal control of the excitation signal

Note that this is Neuman-Neuman configuration with a part which is nondissipated. For stabilization needs to. construct a vector field such that $h \cdot \nu = 0$ **on** Γ_0**.** Can be done when Γ_0 is convex. [Bending the radial vector field on Γ_0 : D. Tataru. I.L. R. Triggiani, X.Zhang.]

Stabilizability -critical case

Theorem (M. Bongarti, J. Rodriguez, I.L DCDS-2022)

Assumptions

- *Let* $\gamma \geq 0$*. Consider* $g(u) = -u$ *in linear dynamics.*
- *Geometric condition of convexity on* Γ_0*.*
- *Initial data:*
  $U(0) = [u(0), u_t(0), u_{tt}(0)] \in H_0 \equiv H^1(\Omega) \times H^1(\Omega) \times L_2(\Omega.)$

Then,

$$||U(t)||_{H_0} \leq C||U(0)||_{H_0} e^{-\omega t}, t > 0$$

Remark: The value of C does not depend on $\gamma \geq 0$. Thus the result is **valid with** $\gamma = 0$ **which is a critical case.**
Other results with the **Dirichlet** data subject to star shaped conditions and no restrictions on Γ_1.. M. Bongarti, I.L, R. Triggiani, .*Applicable Analysis*. 2022.Do not apply to the present configuration.

Stability- nonlinear critical case

Theorem (M. Bongarti, I.L DCDS-S-2022)

Assumptions

- *Let* $\gamma \geq 0$. *Consider* $g = -u$ *and nonlinear dynamics.*
- *Geometric convexity condition of convexity on* Γ_0.
- *Initial data:*
  $U(0) = [u(0), u_t(0), u_{tt}(0)] \in H_1 \equiv H^{2-\epsilon}(\Omega) \times H^1(\Omega) \times L_2(\Omega)$
  subject to compatibility conditions.

$$D_t u(0) + \frac{\partial u(0)}{\partial n} = 0 \text{ on } \Gamma_1, \frac{\partial u(0)}{\partial n} + u(0) = 0 \text{ on } \Gamma_0$$

Then, *there exists* $r > 0$ *such that if* $||U(0)||_{H_0} \leq r$ *then there exists a unique solution such* $U(t) = [u(t), u_t(t), u_{tt}(t)] \in C([0,\infty); H_1)$ *such that*

$$||U(t)||_{H_1} \leq C(||U(0)||_{H_1}) e^{-\omega t}, t > 0$$

Remark: (1) The value of C does not depend on $\gamma \geq 0$. Thus the result is **valid with** $\gamma = 0$ **which is a critical case.** (2) Note the loss of differentiability $\epsilon > 0$. (3) Note smallness required only in H_0.

Difficulty-geometry

Lemma

Energy Identity *Let* $T > 0$*. If* $\Psi = (u, z, z_t)$*,* $z = u_t + \frac{c^2}{b} u$*, is a weak solution then*

$$E_1(T) + \int_t^T D_\Psi(s) ds = E_1(t) + \int_t^T \int_\Omega f(u, u_t) z_t d\Omega ds \qquad (6)$$

holds for $0 \leq t \leq T$*, where* D_Ψ *represents the interior/boundary damping and is given by*

$$D_\Psi := b \int_{\Gamma_1} \kappa_1 z_t^2 d\Gamma_1 + \int_\Omega \gamma u_{tt}^2 d\Omega ds \qquad (7)$$

Reconstruction of total integral energy

$$\int_s^{T-s} E_1(t)dt \lesssim E_1(s) + E_1(T-s) + \int_0^T D_\Psi(s)ds + \int_s^{T-s} \tilde{B}(\Gamma)(t)dt + \int_Q f(u,u_t)^2 dQ + \|z\|^2_{L^2(s,T-s;L^2(\Omega))}. \tag{8}$$

where

$$\tilde{B}(\Gamma) := \frac{1}{2}\int_\Gamma \left(z_t^2 - b|\nabla z|^2\right)(h\cdot\nu)d\Gamma + b\int_\Gamma \partial_\nu z M_h(z)d\Gamma + \int_\Gamma z\partial_\nu z d\Gamma + \int_{\Gamma_0} \kappa_0 |z|^2 d\Gamma_0 \tag{9}$$

BAD GUY:

$$\frac{1}{2}\int_{\Gamma_0} \left(z_t^2 - b|\nabla z|^2\right)(h\cdot\nu)d\Gamma$$

Needs to be "killed". Will be handled by a new geometric construct.

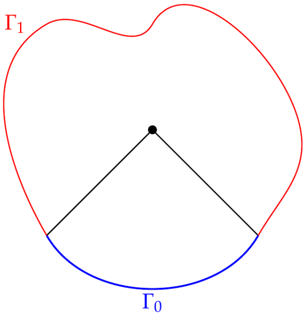

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Figure: Representation of the domain. Needs $\vec{h} \cdot \vec{n} = 0, \ on \ \Gamma_0$

convexity of Γ_0: there exists a vector field
$h(x) = [h_1(x), \cdots, h_d(x)] \in C^2(\overline{\Omega})$ such that

$$h \cdot \nu = 0 \text{ on } \Gamma_0 \tag{10}$$

with ν the unit outward normal, $\delta > 0$ and all vector $v(x) \in [L^2(\Omega)]^n$, we have

$$\int_\Omega J(h)|v(x)|^2 d\Omega \geqslant \delta \int_\Omega |v(x)|^2 d\Omega, \tag{11}$$

The solution to the open loop boundary problem is defined by singular integrals:

$$U(t) = e^{At}U(0) + \begin{bmatrix} 0 \\ 0 \\ b\mathcal{A}N_0 g(t) \end{bmatrix}$$

$$-A\int_0^t e^{A(t-s)} \begin{bmatrix} 0 \\ 0 \\ b\mathcal{A}N_0 g(s) \end{bmatrix} ds - \int_0^t e^{A(t-s)} \begin{bmatrix} 0 \\ 0 \\ c^2\mathcal{A}N_0 g(s) \end{bmatrix}$$

The model is obtained by homogenization of the boundary data and using compatibility conditions. $\mathcal{A} = A_N$, N_0 is. **Neumann harmonic extension**. A priori:

$g \to U$ bounded operator $L_2(\Sigma) \to L_2(D(A^2)')$. We will do better

- **Hidden Neumann regularity.** $H^{2/3}(\Omega)$ [optimal D. Tataru] regularity of u and regularity of $D_\tau u, u_t$:

  $$u \in C(H^{2/3}(\Omega)), u_t \in C(H^{-1/3}(\Omega)), u_{tt} \in C(H^{-4/3}(\Omega)) \oplus L_2(H^{-1/4})$$

Bucci-Eller, Bucci-Pandolfi, IL-Triggiani,.

Theorem

Assume: $g \in L_2(U) = L_2(L_2(\Gamma))$. Ω *is "smooth" .*
Then

- *with* $U(0) = 0$

$$u \in C(H^{2/3}(\Omega)), u_t \in C(H^{-1/3}(\Omega)), u_{tt} \in C(H^{-4/3}(\Omega)) \oplus L_2(H^{-1/4})$$

- *Trace Hidden regularity:*

$$u_t|_\Gamma \in L_2(\Sigma), For\ g = 0, U(0) \in H^{4/3} \times H^{1/3} \times H^{-2/3}$$

Pathology: $u_{tt} \notin C(H^{-4/3}(\Omega))$. It has a component in $L_2(H^{-1/4})$

Input-output dynamics for Optimal Control Problem.

- $y(t) = (u(t), u_t(t), u_{tt}(t)) = e^{At}y_0 + bc^{-2}B_0 g(t) + (Lg)(t)$

$$L(g) \equiv \int_0^t e^{A(t-s} B_0 g(s) + bc^{-2} A \int_0^t e^{A(t-s)} B_0 g(s) ds$$

- control operators $B_0 \in L(L_2(\Gamma_0) \to [D(\mathcal{A})]'), i = 0, 1,$
- Let $g \in L_2(0, \infty; U), U = L_2(\Gamma), \; Y = H^1 \times H^1 \times L_2.$ Minimize

$$J(g) = J(g, y(g)) = \int_0^\infty ||R(y - y_d)||_Y^2 + \int_0^\infty |g|_{L_2(\Gamma)}^2$$

Note: **Operator** B_0 **is a boundary operator-uncloseable.** Very **rough** transient dynamics in $[D(A^2)]'$

Input-output dynamics for Optimal Control Problem with absorbing boundary conditions.

- $y(t) = (u(t), u_t(t), u_{tt}(t)) = e^{At} y_0 + B_1 g(t) + (Lg)(t)$

$$L(g) \equiv \int_0^t e^{A(t-s} B_0 g(s) + A \int_0^t e^{A(t-s)} B_1 g(s) ds$$

- control operators $B_i \in L(L_2(\Gamma_0) \to [D(\mathcal{A})]'), i = 0, 1$, are given by

$$B_0 = \begin{pmatrix} 0 \\ 0 \\ \tau^{-1} c^2 \mathcal{A} N_0 \end{pmatrix}, \qquad B_1 = b c^{-2} B_0 \qquad (12)$$

$$A = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\tau^{-1} c^2 \mathcal{A} & -\tau^{-1} \left[b\mathcal{A} + c\mathcal{A} N_1 N_1^* \mathcal{A} \right] & -\tau^{-1} \left[\alpha I + \frac{b}{c} \mathcal{A} N_1 N_1^* \mathcal{A} \right] \end{pmatrix}$$

Theorem (I.L. R. Triggiani, 2022)

1. **Partial Regularity** *For any* $y_0 \in [D(A^{*2})]'$, $\exists$ *unique optimal* $g^* \in C([0,\infty; U = L_2(\Gamma_0)) : Ry^* \in C[0,\infty; Y]$.
2. **Riccati Equation**. $\exists$ *a selfadjoint positive operator* P *on* $L(Y)$ *s.t.* ::
   - $A^*PA \in L(Y), B_1^*A^*P \in L(Y;U)$

   which satisfies the nonstandard Riccati equation: for all $y, \hat{y} \in Y$

$$((Ay, P\hat{y})_Y + (Py, A\hat{y})_Y + (Ry, R\hat{y})_Y =$$

$$(B_1^*R^*Ry + K_{B_0,B_1}Py, [I + B_1^*R^*RB_1]^{-1}[B_1^*R^*R\hat{y} + K_{B_0,B_1}P\hat{y})_U$$

$$K_{B_0,B_1} \equiv B_0^* + B_1^*A^*$$

3. **Feedback synthesis**: *The optimal control* g^* *satisfies* $\forall t > 0$

$$g^*(t) = -G^{-1}[B_0^* + B_1^*A^*]Py^*(t)$$

where $G \equiv I - [B_0^* + B_1^*A^*]PB_1$ *is* **bounded invertible** *on* U .

COMMENTS

- Improved regularity of **Riccati operator** P.
- **"Gain" operator** $KP = [B_1^* A^* + B_0^*]P$ is bounded. $Y \to U$. Typical for analytic dynamics but not for hyperbolic.
- **Unbounded coefficients** : $B_1^* A^* P$ with B_1 boundary operator.
- **Key element:invertibility of** $I - KPB_1$ on $L_2(\Gamma) = U$. This requires consideration of singular control problem with a parameter $g(0)$. Needs to show the injectivity.

$$[B_0^* + B_1^* A^*]PB_1 v = v \to v \equiv 0$$

Contradict: $v \neq 0$. Consider $-B_1 v$ as the initial datum for the process. Using the theory after some calculations one shows that $g^*(0, -B_1 v) = v$ and y_0 coincides with $y_0 = -B_1 g^*(0, -B_1 v)$ This gives

$$y(t) = -e^{At} B_1 g^*(0) + B_1 g^*(t) + (L_0 g^*)(t), t > 0$$

The optimality implies $g^* \equiv 0$ and $v = 0$. This provides injectivity. Bounded invertiblity uses compactness induced by hidden regularity of dynamic Neumann map.

Conclusions:

- We solved the original HIFU problem with L_2 **rough** controls. The observed quantities Ry are in $C(Y)$. Rough transient dynamics.
- Existence of solutions to **non-standard** Riccati equation
- **Feedback synthesis** -on line control - achieved.

Open Problems

There are several open problems triggered by the work presented.

- Application of infinite horizon feedback control to **nonlinear system**. It is anticipated that local theory for small initial data should emerge. Such feedback should provide stabilizing effect on nonlinear dynamics.
- Extension of the theory to more **general observation** operator. The structure of the problem is important. It is anticipated that some smoothing effect of the observation will be necessary.
- Consider minimization of $||u - u_d||_\Gamma$. Hidden regularity of Neumann Dynamic map is critical.

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