Random lift of set-valued maps and applications

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Introduction

- Mathematical analysis of *multiagent systems* attracted a renewed interest from the applied mathematics community in view of its capability to model many real-life phenomena with a good degree of accuracy.
  The **field of application** of such models ranges
  - from social dynamics to financial markets
  - from big data analysis to life sciences
- Our goal: we study a general framework describing an abstract two-scale mean field evolution.
- Main Motivation: comes from the study of multiagent system, both in mean field control and in the mean field game setting, however some results can be adapted to more general situations

In the presentation of the abstract framework we are going to still use the terms macroscopic and microscopic scale to describe the occurring phenomena. Such a terminology is justified by the application to mean field problems for multiagent systems.

The microscopic scale

We consider three separable metric spaces X, Y, Z, which - still in analogy with the multiagent systems - we are going to call in the following way

- $X \Rightarrow$ space of states;
- $Y \Rightarrow$ space of microscopic evolution;
- $Z \Rightarrow$ space of observable macroscopic evolution.

We will consider then a set-valued map

- $S : Z \times X \rightrightarrows Y \Rightarrow$ which will associate to macroscopic evolution and each state a microscopic evolution.

In a multiagent system interpretation

The set-valued map S associates to each pair $(z, x) \in Z \times X$ the set of admissible microscopic evolutions starting from the initial state $x \in X$ provided that the observable macroscopic evolution is described by $z \in Z$.

Random lift

In many cases, the set-valued map $S : Z \times X \rightrightarrows Y$ turns out to be not adequate to describe the system, as customary in mean-field constructions.

Possible motivations

We want to model situations in which we have only a probabilistic knowledge of the initial state (e.g. noise in the measurements, systems with incomplete information...).

We want to model multi-agent systems, where only a statistical aggregated description of the states is available (e.g. gas/crowd dynamics, financial markets, social dynamics...).

Random lift

Denote by $\mathscr{P}(\mathcal{X})$ the space of Borel probability measures on a metric space $\mathcal{X}$.

Definition (Random Lift)

Starting from the set-valued map $S : Z \times X \rightrightarrows Y$, we define its random lift w.r.t. X to be the set-valued map $\Xi : Z \times \mathscr{P}(X) \rightrightarrows \mathscr{P}(X \times Y)$ defined as

$$\Xi(z, \mu) := \left\{ \eta \in \mathscr{P}(X \times Y) : \eta\left(\operatorname{graph} S^z\right) = 1, \text{ and } \operatorname{pr}^{(1)} \sharp \eta = \mu \right\},$$

where $\operatorname{pr}^{(1)}(x, y) = x$ for all $(x, y) \in X \times Y$ and $S^z : X \rightrightarrows Y$ is the set valued map defined as $S^z(x) = S(z, x)$ for all $x \in X$, $z \in Z$.

We will also write $\Xi^z(\mu) = \Xi(z, \mu)$.

Random lift - interpretation

Given $\boldsymbol{\eta} \in \Xi^z(\mu)$, for $\boldsymbol{\eta}$-a.e. $(x, y) \in X \times Y$ we have that $y \in S^z(x)$, thus the measure $\boldsymbol{\eta}$ charges mass only on the subsets of admissible microscopic evolutions under macroscopic observable evolution described by z.

According to the disintegration theorem, we write $\boldsymbol{\eta} = (\mathrm{pr}^{(1)}\sharp\boldsymbol{\eta}) \otimes \eta_x = \mu \otimes \eta_x$, where $\{\eta_x\}_{x\in X}$ is a Borel family of probability measures, uniquely defined for μ-a.e. $x \in X$, and $\mathrm{pr}^{(1)}\sharp\boldsymbol{\eta}$ is the pushforward measure[1] of $\boldsymbol{\eta}$ w.r.t. the Borel map $\mathrm{pr}^{(1)}$.

Informally

- **the measure μ =** the weights assigned to the initial conditions in X,
- **the measure η_x =** a probability measure concentrated on the set $S^z(x)$ of the admissible evolutions starting from x which describes how the mass initially present at x is splitted on the evolutions from x under the macroscopic observable evolution z.

[1] In general if $\theta \in \mathscr{P}(\mathcal{X})$ and $r : \mathcal{X} \to \mathcal{Y}$ is a Borel map, we have $r\sharp\theta \in \mathcal{P}(\mathcal{Y})$ and $(r\sharp\theta)(B_{\mathcal{Y}}) = \theta(r^{-1}(B_{\mathcal{Y}}))$ for every Borel subset $B_{\mathcal{Y}}$ of $\mathcal{Y}$.

The macroscopic scale

After having constructed the random lift $\Xi : Z \times \mathscr{P}(X) \rightrightarrows \mathscr{P}(X \times Y)$, which describes the admissible evolutions in a probabilistic form, we introduce

- an evaluation map $\mathscr{E}_p : \mathscr{P}_p(X \times Y) \to Z$, which plays the same role of an observability relation, expressing which information from the probabilistic evolutions can be actually catched from the system and affects the microscopic evolution.
- the set-valued map $\mathbf{Y} : Z \times \mathscr{P}_p(X) \rightrightarrows Z$ by

$$\mathbf{Y}(z, \mu) = \mathscr{E}_p(\Xi(z, \mu)),$$

and write $\mathbf{Y}^z(\mu) = \mathbf{Y}(z, \mu)$. The set $\mathbf{Y}^z(\mu)$ describes the observable macroscopic evolutions strarting from μ under the observable macroscopic evolution described by z.

Admissible observable macroscopical trajectories

Given a measure $\mu \in \mathscr{P}_p(X)$, we built the set-valued map $\mathbf{Y}(\cdot,\mu) : Z \rightrightarrows Z$. We define the set of admissible observable macroscopical trajectories starting from μ as the fixed point of the above map, i.e.,

$$\mathscr{A}(\mu) := \{z \in Z : \; z \in \mathbf{Y}(z,\mu)\}.$$

Indeed, there is coherence between the (weighted) microscopic evolution and the macroscopic observed evolution if and only $z \in \mathbf{Y}(z,\mu)$.

The set-valued map $\mathscr{A} : \mathscr{P}_p(X) \rightrightarrows Z$ expresses all the admissible macroscopic observable evolution with respect to the weigth assigned on the initial state.

Natural problems and aims of the investigation

The construction so far outlined leads to the following natural question, which basically constitute the aim of our investigation.

1. Which properties on the set-valued map $S : Z \times X \rightrightarrows Y$ ensures that $\mathscr{A}(\mu) \neq \emptyset$?

2. Which properties on the set-valued map $S : Z \times X \rightrightarrows Y$ (such that closedness of the graph, compactness of the images...) are transferred to the set-valued map of the fixed point $\mathscr{A} : \mathscr{P}_p(X) \rightrightarrows Z$?

3. Derive special results for the multiagent systems evolving in an underlying finite-dimensional space.

First properties of the random lift[2]

1. $\Xi(\cdot)$ has always convex images w.r.t. the linear structure of $(C_b^0(X \times Y))'$, even if $S(\cdot)$ has not convex images. If Z is a convex subset of a linear space and $\mathscr{E}$ is an affine evaluation, we have that $\Upsilon(\cdot)$ has convex or empty images.
2. $S(\cdot)$ has closed graph if and only if $\Xi(\cdot)$ has closed graph;
3. given $z \in Z$, suppose that for every compact $K \subseteq X$ the set $\mathrm{graph}\left(S^z_{|K}\right)$ is compact in $X \times Y$. Then for every relative compact $\mathscr{K} \subseteq \mathscr{P}(X)$, the set $\Xi^z(\mathscr{K}) := \bigcup_{\mu \in \mathscr{K}} \Xi^z(\mu)$ is relative compact.

[2] C., R., Marigonda, A., Mogentale, M., Random lifting of set-valued maps, **accepted** to appear In Large-scale scientific computing, Springer,2022.
C., Marigonda, Ricciardi, Random lift of set-valued maps and applications to multiagent dynamics, **submitted**

Towards fixed point arguments

- The previous results provided us with the basic topological structure needed to apply well-known fixed point theorems for set-valued map (e.g. Kakutani-Fan-Glicksberg fixed point, or Nadler's fixed point) to obtain the nonemptyness of the set of admissible observable macroscopic evolutions.
- The assumptions are established mainly on the set-valued map $S(\cdot)$ expressing the microscopic admissible evolution (the others will be related to the structure of the spaces).

Fixed point Theorem [3]

Theorem (C.-Marigonda-Ricciardi)

Let X be a compact metric space. Let $p \geq 1$, and Z be a subset of a linear space. Suppose that

(i) $(z, x) \mapsto S^z(x)$ *is upper semicontinuous with nonempty compact images;*

(ii) $\mathscr{E}_p$ *is a continuous affine evaluation;*

(iii) *there exists a compact convex set* $K \subseteq Z$ *such that* $\Upsilon^z_{\mathscr{E}_p}(\mu) \subseteq K$ *for all* $z \in K$, $\mu \in \mathscr{P}_p(X)$;

(iv) *if we endow* K *with the topology inherited by* Z*, we have that*
- *every point of* K *has a base of convex (not necessarily open) neighborhoods,*
- *the map* $K \times K \times [0, 1] \to K$ *defined by* $(k_1, k_2, \lambda) \mapsto \lambda k_1 + (1 - \lambda)k_2$ *is continuous.*

Then

1. $\mathscr{A}(\mu) \neq \emptyset$ *for all* $\mu \in \mathscr{P}_p(X)$
2. $\mathscr{A}$ *is upper semicontinuous.*

[3] C., Marigonda, Ricciardi, Random lift of set-valued maps and applications to multiagent dynamics, **submitted**

APPLICATION TO CONSTRAINTS MEAN FIELD GAMES

Constrained Mean Field Games problem

OUR GOAL: To study a MFG problem concerning agents aiming to minimize a (nonlocal) integral functional, but also constrained to move on a the closure of an open bounded $\overline{\Omega} \subset \mathbb{R}^d$

$$\boxed{\gamma(t) \in \overline{\Omega} \quad \forall t \in [a, b]} \quad \text{(STATE CONSTRAINT)}$$

where $\gamma(t) =$ position of player at time t.

MFG with state constraints appear very naturally in the applications:

- macroeconomic models $\Rightarrow$ so-called heterogeneous agent models, see for istance, Huggett's model of income and wealth distribution[4].
- pedestrians models $\Rightarrow$ to explain the behavior of crowds, to regulate traffic, see for instance[5]

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[4] Achdou, Y., Buera, F. J., Lasry, J.-M., Lions, P.-L., and Moll, B. *Partial differential equation models in macroeconomics*, Philosophical Transactions of the Royal Society A, 372 (2028):20130397, 2014.
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[5] Carlini, E., Festa, A., Silva, F. J., and Wolfram, M.-T., *A semi-lagrangian scheme for a modified version of the Hughes' model for pedestrian flow*, Dynamic Games and Applications, 7(4):683-705, 2017.
Cristiani, E., Priuli, F. S., and Tosin, A., *Modeling rationality to control self-organization of crowds: an environmental approach*, SIAM Journal on Applied Mathematics, 75(2):605-629, 2015.

Constrained Mean Field Games problem

- **Between 2018 and 2021**[6], Cannarsa, C. and Cardaliaguet studied a mean field games problem in presence of state constraints
- **Main difficulties**: The presence of a boundary in the domain makes difficult to write a MFG system, in the sense that the Hamiltonian must be carefully defined near to the boundary. Furthermore, phenomena like concentration of agents may occurr.
- **Main idea to overcome difficulties**: to introduce the notion of Constrained Mean Field Games Equilibria.
  Roughly speaking, a measure $\boldsymbol{\eta}$ on curves is a CMFGE if it is concentrated on curves γ minimizing the functional evaluated on $\boldsymbol{\eta}$ and γ.

[6] Cannarsa, P., C.,, Existence and uniqueness for Mean Field Games with state constraints,"PDE models for multi-agent phenomena", P. Cardaliaguet, A.Porretta, F. Salvarani editors, Springer INdAM Series, 2018
Cannarsa, P.,C., and Cardaliaguet, P.,$C^{1,1}$–smoothness of constrained solutions in the calculus of variations with application to mean field games,Mathematics in Engineering,1(1): 174-203, doi:10.3934/Mine.2018.1.174, 2018
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CMFGE as Fixed Point of Random Lifting of Set-Valued Maps

Let $I = [a, b] \subseteq \mathbb{R}$ be a compact interval, X be a finite-dimensional real space, $\Omega \subseteq X$ be open and bounded.

Consider

- a set-valued map $F : I \times \mathscr{P}(X) \times X \rightrightarrows X$ expressing the microscopical dynamics of the agents
- an exit cost $g : \mathscr{P}(X) \times X \to [0, +\infty[$
- the cost for each agent, provided that the macroscopical evolution is described by $\boldsymbol{\theta} \in \mathscr{P}(C^0(I; X))$, is given by a current cost $\ell : I \times \mathscr{P}(X) \times X \times X \to [0, +\infty]$, as follows

$$J^{\boldsymbol{\theta}}[\gamma] = \begin{cases} \displaystyle\int_a^b \left[\lambda I_{F(t, e_t\sharp\boldsymbol{\theta}, \gamma(t))}(\dot\gamma(t)) + I_{\overline{\Omega}}(\gamma(t)) + \ell(t, e_t\sharp\boldsymbol{\theta}, \gamma(t), \dot\gamma(t))\right] dt + g(e_T\sharp\boldsymbol{\theta}, \gamma(T)), & \text{if } \gamma \in AC(I; X); \\ \\ +\infty, \text{ otherwise}, \end{cases}$$

where $e_t(\gamma) = \gamma(t)$ for every $\gamma \in C^0(I; X)$, $\lambda \in \{0, 1\}$, and we use the convention that $\lambda \cdot +\infty = 0$ if $\lambda = 0$.

When the macroscopical evolution is described by $\theta \in \mathscr{P}(C^0(I; X))$, to have a finite cost each agent must respect the state constraint and stay in $\overline{\Omega}$ and, if $\lambda = 1$, he/she is allowed to move only according the trajectories of the differential inclusion.

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Fitting in the abstract framework

- Choose $Y = C^0(I;X)$ and $Z = \mathscr{P}(Y)$,
- Introduce $S_I : Z \times X \rightrightarrows Y$ by

$$S_I(\boldsymbol{\theta}, x) = \left\{ \hat{\gamma} \in Y : \hat{\gamma}(a) = x, J^{\boldsymbol{\theta}}[\hat{\gamma}] < +\infty \text{ and } J^{\boldsymbol{\theta}}[\hat{\gamma}] = \inf_{\substack{\gamma \in AC(I;X) \\ \gamma(a)=x}} J^{\boldsymbol{\theta}}[\gamma] \right\}.$$

- The Random lift $\Xi_{\boldsymbol{I}} : Z \times \mathscr{P}(X) \rightrightarrows \mathscr{P}(X \times Y)$ is

$$\Xi_{\boldsymbol{I}}(\boldsymbol{\theta}, \mu) = \left\{ \boldsymbol{\eta} \in \mathscr{P}(X \times Y) : \boldsymbol{\eta}(\text{graph } S_I^{J,\boldsymbol{\theta}}) = 1 \text{ and } \mathrm{pr}_1 \sharp \boldsymbol{\eta} = \mu \right\}.$$

- Define the evaluation map $\mathscr{E}_p : \mathscr{P}_p(X \times Y) \rightrightarrows Z$ for the random lift $\Xi_{\boldsymbol{I}}$ by

$$\mathscr{E}_p(\boldsymbol{\eta}) = \mathrm{pr}_2 \sharp \boldsymbol{\eta}$$

- $\Upsilon_I : Z \times \mathscr{P}_p(X) \rightrightarrows Z$ is defined by $\mathscr{E}_p(\Xi_I^{J,\boldsymbol{\theta}}(\mu)) = \Upsilon_I(\boldsymbol{\theta}, \mu)$
- Define $\mathscr{A}_I : \mathscr{P}_p(X) \rightrightarrows Z$ to be the set-valued map given by

$$\mathscr{A}_I(\mu) := \{\boldsymbol{\theta} \in Z : \boldsymbol{\theta} \in \Upsilon_I^{J,\boldsymbol{\theta}}(\mu)\},$$

i.e., $\mathscr{A}_I(\mu)$ is the set of fixed points of $\boldsymbol{\theta} \mapsto \Upsilon_I^{J,\boldsymbol{\theta}}(\mu)$.

CMFGA as Fixed Point of Random Lifting of Set-Valued Maps[7]

Theorem (C.-Marigonda-Ricciardi)

Let $\mu \in \mathscr{P}(X)$ be satisfying $\mu(\overline{\Omega}) = 1$. Suppose that

- *F and $\overline{\Omega}$ enjoy the following weak invariance property:*

  (VP) *Defined $\tilde{F} : I \times \mathscr{P}(X) \times X \rightrightarrows \mathbb{R} \times X$ by $\tilde{F}(t,\theta,x) = \{1\} \times F(t,\theta,x)$, for any $(t_0,x) \in I \times \overline{\Omega}$ and any $\boldsymbol{\theta} \in \mathscr{P}(C^0([t_0,b];X))$ such that $\gamma(t) \in \overline{\Omega}$ for $\boldsymbol{\theta}$-a.e. $\gamma \in C^0([t_0,b];X)$ and all $t \in [t_0,b]$, there exists a solution $t \mapsto \tilde{\gamma}(t) := (\gamma_o(t),\gamma(t))$ on $[t_0,b]$ of the differential inclusion $\dot{\tilde{\gamma}}(t) \in \tilde{F}(t, e_t\sharp\boldsymbol{\theta}, \tilde{\gamma}(t))$ satisfying $\gamma(t_0) = x$ and $\gamma(t) \in \overline{\Omega}$ for all $t \in [t_0,b]$, where $e_t(\gamma) = \gamma(t)$ for every $\gamma \in C^0(I;X)$.*

- *The current cost $\ell(\cdot)$ satisfies*

  (L1) *$t \mapsto \ell(t,\mu,x,v)$ is $\mathrm{Leb}(I) \otimes \mathrm{Bor}(\mathscr{P}(X) \times X \times X)$-measurable*

  (L2) *$\ell(t,\cdot,\cdot,\cdot)$ is lower semicontinuous for a.e. $t \in I$;*

  (L3) *there exist constants $c_1, c_0 > 0$ such that for all $(t,\mu,x,v) \in I \times \mathscr{P}(X) \times X \times X$, it holds*

  $$\ell(t,\mu,x,v) \geq c_1|v|^2 - c_0, \quad \forall (t,\mu,x,v) \in I \times \mathscr{P}(X) \times X \times X,$$

  (L4) *$v \longmapsto \ell(t,\mu,x,v)$ is convex for all $(t,\mu,x) \in I \times \mathscr{P}(X) \times X$.*

- *The exit cost $g(\cdot)$ is lower semicontinuous.*

- *If $\lambda = 1$ we assume that F is compact, convex valued, measurable in t and continuous in the other variables.*

Then $\mathscr{A}_I(\mu) \neq \emptyset$.

[7] C., R., Marigonda, A., Constrained Mean Field Games Equilibria as Fixed Point of Random Lifting of Set-Valued Maps, **accepted**, 2022
C., Marigonda, Ricciardi, Random lift of set-valued maps and applications to multiagent dynamics, **submitted**

Some Remarks

- Given $\boldsymbol{\theta} \in \mathscr{A}_I(\mu)$, let $\boldsymbol{\eta} \in \Xi_I^{J,\boldsymbol{\theta}}(\mu)$ be such that $\mathscr{E}_p(\boldsymbol{\eta}) = \boldsymbol{\theta}$. By disintegrating $\boldsymbol{\eta}$ w.r.t. the Borel map e_a, we have $\boldsymbol{\eta} = \mu \otimes \eta_x$, where $\{\eta_x\}_{x \in X}$ is a Borel family of probability measures, uniquely defined μ-a.e. and for μ-a.e. $x \in X$ and η_x-a.e. $(z, \gamma) \in X \times Y$, we have $(z, \gamma) \in \operatorname{graph} S_I^{J,\boldsymbol{\theta}}$ with $z = x$, thus $\gamma(a) = x$ and γ is a minimizer of $J^{\boldsymbol{\theta}}(\cdot)$ among all $\xi(\cdot) \in AC(I; X)$ satisfying $\xi(a) = x$. Since $\boldsymbol{\theta} = \mathrm{pr}_2 \sharp \boldsymbol{\eta}$, we conclude that for $\boldsymbol{\theta}$-a.e. $\gamma \in Y$ we have that γ is a minimizer of $J^{\boldsymbol{\theta}}$ among all the absolutely continuous curves sharing its initial condition.
- If $\lambda = 0 \Rightarrow \boldsymbol{\theta}$ is a constrained MFG equilibrium in the sense of Cannarsa-C.-Cardaliaguet.

Remarks and further development

The abstract framework proposed can be applied to any $S_I(x, \theta)$ assigning to each point of X a set of trajectories depending on θ and satisfying some properties, so not only the solution map of differential inclusions, or minimizers of functionals, even though these two cases are the main motivating examples.

Among the most relevant cases that we plan to investigate, we have

- solutions of parametric time-depending maximal monotone differential inclusions, e.g., $\dot{\gamma}(t) = -\partial V(t, \mu_t, \gamma(t))$, $\gamma(0) = x$. In particular, we believe that results on this can be useful to study problems related to existence and well-posedness of *gradient flows* for functional in the space of probability measures. Some work in this sense is currently in development by Cavagnari-Savaré-Sodini.
- generalized characteristics for some θ-dependent HJ equation, e.g. of the kind $\partial_t u(t, x) + H(t, x, Du(t, x), \theta) = 0$. In this case, the similarity with some basic feature appearing in first-order MFG needs to be better understood.

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THANK YOU FOR YOUR ATTENTION