Approximate control for the nonlinear Schrödinger equation on a torus via bilinear controls

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Let us consider the **nonlinear Schrödinger equation** on a torus $\mathbb{T}^n$:

$$i\partial_t\psi(t) = -\Delta\psi(t) + \underbrace{\boxed{k\left|\psi(t)\right|^{2p}\psi(t)}}_{\text{nonlinear term}} + \underbrace{\boxed{\Big(\sum_{j=1}^{q} u_j(t)\mu_j\Big)}}_{\text{bilinear controls}}\psi(t). \qquad \text{(NSE)}$$

- The parameters $k \in \mathbb{R}$ and $p, q \in \mathbb{N}^*$.
- $(\mu_1, ..., \mu_q) \in C^\infty(\mathbb{T}^n)$ and $u = (u_1, ..., u_q) \in L^2(0, T)$ is the **control**.
- The dynamics is **locally well-posed** in H^s with $s \geqslant \frac{n+2}{2}$. When it exists,

$$\psi\big(\text{time} = t; \text{initial state} = \psi_0, \text{control} = u\big)$$

is the solution at time t and with initial state ψ_0 and control u.

Goal: For any ψ_1 and ψ_2 in a suitable space, there exist T and u so that the solution $\psi(T; \psi_1, u)$ **reaches approximately** ψ_2.

Some existing results:

- **Local exact controllability** for the linear dynamics ($k = 0$) in $(0, 1)$.
  - Beauchard, Coron, D., Laurent, Morancey, Nersesyan...
- **Local exact controllability** for the nonlinear dynamics in $(0, 1)$.
  - In presence of Neumann boundaries: Beauchard, Laurent.
  - In presence of Dirichlet boundaries: D, Nersesyan (preprint).
- **Global approximate controllability** for the linear dynamics ($k = 0$).
  - Galerkin approximation or adiabatic theory: Boscain, Boussaïd, Caponigro, Chambrion, Chittaro, Gauthier, Mason, Rossi, Sigalotti...
  - Lyapunov techniques: Mirrahimi, Nersesyan...

Let us consider the nonlinear Schrödinger equation (NSE) when $p = 1$:

$$i\partial_t\psi(t) = -\Delta\psi(t) + k\big|\psi(t)\big|^2\psi(t) + \Big(\sum_{j=1}^{q} u_j(t)\mu_j\Big)\psi(t). \qquad \text{(NSE)}$$

For every vector $l \in \mathbb{Z}^n$, we denote by ϕ_l the **eigenmode** of $-\Delta$:

$$\phi_l(x) = (2\pi)^{-n/2}e^{i\langle x,l\rangle}, \qquad x \in \mathbb{T}^n.$$

Let $A = \{(1,0,...,0),(0,1,...,0),...,(0,0,...,1,0),(1,...,1)\}$.

Assumption. Let us assume that the following property is verified.

(H) $\big\{1, \sin\langle x,a\rangle, \cos\langle x,a\rangle : a \in A\big\} \subset span\{\mu_j : j = 1,...,q\}$.

Remark: The assumption **(H)** is used in a **saturation argument** inspired by a theory developed by Agrachev and Sarychev for additive controls.

Main Theorem1 (D., Nersesyan; preprint - accepted wih minor revisions)

Assume **(H)** *being verified. The nonlinear Schrödinger equation* (NSE) *is* **approximately controllable** *between eigenmodes in any time:*

- *for any* **error** $\epsilon > 0$*, for any couple of* **eigenmodes** ϕ_l *and* ϕ_m *and for any* **time** $T > 0$*, there exists a* **control** $u \in L^2((0,T),\mathbb{R}^q)$ *such that*

$$\|\psi(\mathit{time} = T; \mathit{initial\ state} = \phi_l, \mathit{control} = u) - \phi_m\|_{L^2} < \epsilon.$$

The main **novelties** contained in Main Theorem 1 are the following.

- The controllability is ensured despite the **nonlinear evolution**.
- The control can be performed in **very small time**.

The existing theories consider the linear case and long controls.

Main Theorem 1 is a consequence of the following more general result.

Main Theorem 2 (D., Nersesyan; preprint - accepted wih minor revisions)

Assume **(H)** *being verified. For any* $\epsilon > 0$, ψ_0 *in* $H^s(\mathbb{T}^n)$, $\theta \in C^\infty(\mathbb{T}^n)$ *and* $T > 0$, *there exist a* **time** $0 < \tau \leqslant T$ *and a* **control** u *such that*

$$\left\|\psi(\textit{time} = \tau; \textit{initial state} = \psi_0, \textit{control} = u) - e^{i\theta}\psi_0\right\|_{H^s} < \epsilon. \qquad (2.1)$$

- Let $\psi_0 = \phi_l$. We consider a smooth periodic function θ such that

$$\|e^{i\theta}\phi_l - \phi_m\|_{L^2} < \epsilon/2 \qquad (\text{ex. } \phi_2 = e^{ix}\phi_1,\ n = 1).$$

- From (2.1), $\exists \tau \in (0, T]$ and $\tilde{u}$ such that $\|\psi(\tau; \phi_l, \tilde{u}) - e^{i\theta}\phi_l\|_{H^s} < \frac{\epsilon}{2}$ and:

$$\|\psi(\tau; \phi_l, \tilde{u}) - \phi_m\|_{L^2} \lesssim \|\psi(\tau; \phi_l, \tilde{u}) - e^{i\theta}\phi_l\|_{H^s} + \|e^{i\theta}\phi_l - \phi_m\|_{L^2} < \epsilon.$$

- $\exists \hat{u}$ such that $\psi(T - \tau; \phi_l, \hat{u}) = \phi_l$ and, for $u := \hat{u}\mathcal{X}_{(0,T-\tau]} + \tilde{u}\mathcal{X}_{(T-\tau,T)}$,

$$\|\psi(T; \phi_l, u) - \phi_m\|_{L^2} = \|\psi(\tau; \phi_l, \tilde{u}) - \phi_m\|_{L^2} < \epsilon.$$

We prove **Main Theorem 2** by ensuring the existence of u such that:

$$\psi\Big(\text{time} = \delta;\ \text{initial state} = \psi_0,\ \text{control} = \frac{u}{\delta}\Big) \xrightarrow[H^s]{\delta\to 0} e^{i\theta}\psi_0. \tag{3.1}$$

- Let $\mathscr{H}_{-1} = \{0\}$, $\mathscr{H}_0 = span\{1, \sin\langle x, a\rangle, \cos\langle x, a\rangle : a \in A\}$ and

$$\mathscr{H}_{k+1} = \Big\{\varphi_0 - \sum_{j=1}^{n}(\partial_{x_j}\varphi_1)^2 - \sum_{j=1}^{n}(\partial_{x_j}\varphi_2)^2 - \dots - \sum_{j=1}^{n}(\partial_{x_j}\varphi_d)^2 : \\ \varphi_0, \dots, \varphi_d \in \mathscr{H}_k,\ \ d \in \mathbb{N}^*\Big\}, \qquad k \in \mathbb{N}.$$

- For every $k \in \mathbb{N}$ and $\theta_k \in \mathscr{H}_k$, there exist u_k and $\varphi_{k-1} \in \mathscr{H}_{k-1}$ so that

$$\psi\Big(\delta; e^{i\varphi_{k-1}}\psi_0, \frac{u_k}{\delta}\Big) \xrightarrow[H^s]{\delta\to 0} e^{i\theta_k}\psi_0.$$

- The result follows as $\bigcup_{k=0}^{\infty}\mathscr{H}_k$ is dense in C^{s+2} (**saturation argument**).

Thank you for your attention!