Corso di laurea in Chimica
ESONERO di
ISTITUZIONI di MATEMATICHE I

9 dicembre 2025

1. Risolvere la seguente equazione nel campo complesso

$$2z^2 + 2\bar{z} + \left(\sqrt{3} - i\right)^2 z = 2\left(\frac{\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}}{\cos\frac{\pi}{\cancel{5}10} + i\sin\frac{\pi}{10}}\right)^{20}.$$

2. Determinare il dominio delle seguenti funzioni

$$y = \left(\sqrt{|\sin x - \cos x|} - 1\right)^{-\pi},$$

$$y = \ln\left(\lg_{0,5}\left(x^3 - x\right) - \lg_{0,5}\left|x^3 - 1\right|\right),$$

$$y = \sqrt{\frac{\lg_2^2 x^2 + \lg_2|x| - 3}{\sqrt{x^2 - 1} - 2x}}$$

3. Studiare al variare del parametro reale α la continuità delle seguenti funzioni

$$f(x) = \begin{cases} 2 + \frac{x^2 - 2x}{\sqrt{\tan^2(x-2)}} & x < 2 \\ \alpha & x = 2 \\ (x-2)\sin\frac{1}{x-2} & x > 2 \end{cases}$$

$$g(x) = \begin{cases} \left(\frac{3^x + 1}{2}\right)^{\frac{1}{x}} & x < 0 \\ \alpha & x = 0 \\ \ln\frac{x^2 + 3x}{\sin x^2} & 0 < x < \pi \end{cases}$$

Svolgimento

1. Posto $z = a+ib$ si ha

$z^2 = a^2 - b^2 + 2abi$ $\quad \bar{z} = a - ib \quad (\sqrt{3}-i)^2 z = (3-1-2\sqrt{3}i)(a+ib) =$

$2(1-\sqrt{3}i)(a+ib) = 2(a+\sqrt{3}b+ib-\sqrt{3}ai)$

Poiché $\left(\dfrac{\cos\frac{\pi}{5} + i \operatorname{sen}\frac{\pi}{5}}{\cos\frac{\pi}{10} + i\operatorname{sen}\frac{\pi}{10}} \right)^{20} = \cos 20\left(\frac{\pi}{5}-\frac{\pi}{10}\right) + i \operatorname{sen} 20\left(\frac{\pi}{5}-\frac{\pi}{10}\right) = \cos 2\pi + i \operatorname{sen} 2\pi = 1$

l'equazione diventa

$\cancel{2}(a^2-b^2+2abi) + \cancel{2}(a-ib) + \cancel{2}(a+\sqrt{3}b+bi-\sqrt{3}ai) = \cancel{2}^{\,1}$

$a^2 - b^2 + 2abi + 2a\overset{(+\sqrt{3}b)}{} \cancel{-ib} + \cancel{ib} - \sqrt{3}ai = 1 \iff$

$\begin{cases} a^2 - b^2 + 2a + \sqrt{3}b = 1 \\ 2ab - \sqrt{3}a = 0 \end{cases} \to a(2b-\sqrt{3}) = 0 \to a = 0 \lor b = \frac{\sqrt{3}}{2}$

$\begin{cases} a = 0 \\ -b^2 + \sqrt{3}b = 1 \end{cases} \to b^2 - \sqrt{3}b + 1 = 0 \quad \Delta < 0 \quad \emptyset$ sistema impossibile

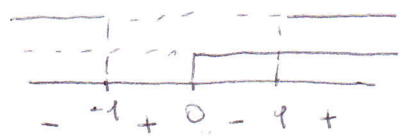
$\begin{cases} b = \frac{\sqrt{3}}{2} \\ a^2 - \frac{3}{4} + 2a + \frac{3}{2} = 1 \end{cases} \to a^2 + 2a - \frac{1}{4} = 0 \qquad \frac{\Delta}{4} = 1 + \frac{1}{4} = \frac{5}{4}$

$a_{1/2} = -1 \pm \frac{\sqrt{5}}{2}$ L'equazione ha 2 soluzioni $-1 \pm \frac{\sqrt{5}}{2} + i\frac{\sqrt{3}}{2}$.

2. $\lg_{0,5}(x^3 - x) - \lg_{0,5}|x^3-1| > 0 \Rightarrow \lg_{0,5}(x^3-x) > \lg_{0,5}(x^3-1) \Rightarrow$

$\begin{cases} x^3 - x > 0 \\ |x^3-1| > 0 \\ |x^3-1| > x^3 - x \end{cases} \qquad \begin{cases} x(x^2-1) > 0 \\ x^3 \neq 1 \\ |x^3-1| > x^3 - x \end{cases}$

$x(x^2-1) > 0 \qquad x > 0 \qquad x < -1 \lor x > 1$



$-1 < x < 0 \lor x > 1$

$|x^3 - 1| > x^3 - x \iff \begin{cases} x^3 - 1 \geq 0 \\ \cancel{x^3} - 1 > \cancel{x^3} - x \end{cases} \lor \begin{cases} x^3 - 1 < 0 \\ -x^3 + 1 > x^3 - x \end{cases} \iff \begin{cases} x \geq 1 \\ x > 1 \end{cases} \lor \begin{cases} x < 1 \\ 2x^3 - x - 1 < 0 \end{cases}$

$x > 1 \lor \begin{cases} x < 1 \\ (x-1)(\underbrace{2x^2 + 2x + 1}_{>0\ \forall x \text{ poiché } \Delta < 0}) < 0 \end{cases} \to x < 1$

	2	0	-1	-1
1		2	2	1
	2	2	1	//

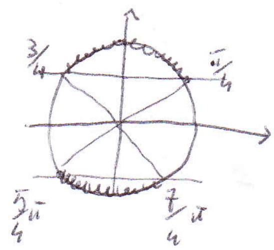
$\begin{cases} -1 < x < 0 \lor x > 1 \\ x > 1 \lor x < 1 \end{cases}$ ~~[illegible]~~ $D =]-1, 0[\cup]1, +\infty)$

- $\sqrt{|\operatorname{sen} x - \cos x|} - 1 > 0 \iff \sqrt{|\operatorname{sen} x - \cos x|} > 1 \iff |\operatorname{sen} x - \cos x| > 1$

Poiché $\operatorname{sen} x - \cos x = \operatorname{sen} x - \operatorname{tg}\frac{\pi}{4}\cdot\cos x = \dfrac{\cos\frac{\pi}{4}\operatorname{sen} x - \operatorname{sen}\frac{\pi}{4}\cos x}{\cos\frac{\pi}{4}} = \dfrac{\operatorname{sen}\left(x-\frac{\pi}{4}\right)}{\cos\frac{\pi}{4}}$

la disequazione diventa

$\left|\operatorname{sen}\left(x-\frac{\pi}{4}\right)\right| > \frac{\sqrt{2}}{2} \iff \operatorname{sen}\left(x-\frac{\pi}{4}\right) < -\frac{\sqrt{2}}{2} \vee \operatorname{sen}\left(x-\frac{\pi}{4}\right) > \frac{\sqrt{2}}{2}$



$\iff \frac{\pi}{4} + 2k\pi < x - \frac{\pi}{4} < \frac{3}{4}\pi + 2k\pi \vee \frac{5}{4}\pi + 2k\pi < x - \frac{\pi}{4} < \frac{7}{4}\pi + 2k\pi$

$\iff \frac{\pi}{2} + 2k\pi < x < \pi + 2k\pi \vee \frac{3}{2}\pi + 2k\pi < x < 2\pi + 2k\pi \qquad k \in \mathbb{Z}$

$ID = \bigcup_{k\in\mathbb{Z}} \left]\frac{\pi}{2} + 2k\pi, \pi + 2k\pi\right[\cup \left]\frac{3}{2}\pi + 2k\pi, 2\pi + 2k\pi\right[$

- $\begin{cases} \dfrac{\lg_2^2 x^2 + \lg_2|x| - 3}{\sqrt{x^2-1} - 2x} \geq 0 \\ x^2 - 1 \geq 0 \end{cases}$ $\rightarrow$ applico la regola del segno quando $x \leq -1 \vee x \geq 1$

$x \leq -1 \vee x \geq 1$

$N \geq 0 \quad (\lg_2 x^2)^2 + \lg_2|x| - 3 \geq 0$

$4\lg_2^2|x| + \lg_2|x| - 3 \geq 0$ posto $\lg_2|x| = t$

$4t^2 + t - 3 \geq 0 \qquad t_{1,2} = \dfrac{-3\pm 7}{8} = \begin{cases} -1 \\ \frac{3}{4} \end{cases}$

$t \leq -1 \vee t \geq \frac{3}{4}$

$\lg_2|x| \leq -1 \vee \lg_2|x| \geq \frac{3}{4} \Rightarrow 0 < |x| \leq \frac{1}{2} \vee |x| \geq 2^{3/4} \Rightarrow$

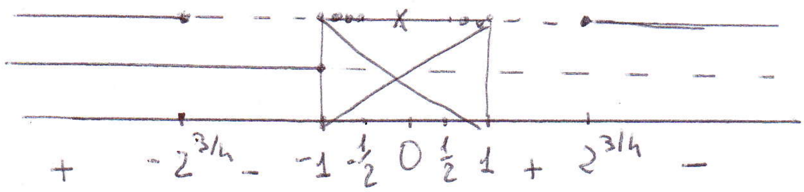
$-\frac{1}{2} < x < \frac{1}{2} \wedge x \neq 0 \quad \vee \quad x \leq -2^{3/4} \vee x \geq 2^{3/4}$

$D > 0 \quad \sqrt{x^2-1} > 2x \iff \begin{cases} x < 0 \\ x^2 - 1 \geq 0 \end{cases} \vee \begin{cases} x > 0 \\ x^2 - 1 > 4x^2 \end{cases} \iff$

$\begin{cases} x \leq 0 \\ x \leq -1 \vee x \geq 1 \end{cases} \vee \begin{cases} x > 0 \\ 3x^2 < -1 \end{cases} \iff x \leq -1$

$x \leq -1 \qquad \emptyset$

$ID = (-\infty, -2^{3/4}] \cup [1, 2^{3/4}]$



- Poiché

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2 + \frac{x^2-2x}{\sqrt{tg^2(x-2)}} = \lim_{x\to 2^-} 2 + \frac{x(x-2)}{|tg(x-2)|} = \lim_{x\to 2^-} 2 + x\,\frac{x-2}{-tg(x-2)}$$

$tg(x-2)<0$ per $x<2$

$$= 2-2=0 \qquad \left(\lim_{x\to 2}\frac{x-2}{tg(x-2)} \overset{y=x-2}{=} \lim_{y\to 0}\frac{y}{tg\,y} = 1\right)$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \underbrace{(x-2)}_{\to 0}\, \underbrace{sen\frac{1}{x-2}}_{\text{limitata}} = 0$$

perché prodotto di funzione infinitesima per una limitata

$\Rightarrow \exists \lim_{x\to 0} f(x) = 0$ Segue che se $\alpha=0$ $\lim_{x\to 0} f(x) = 0 = f(0)$ f continua

se $\alpha\neq 0$ $\lim_{x\to 0} f(x) = 0 \neq f(0)$ f ha discont. eliminabile in $x=0$

- Poiché

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{3^x+1}{2}\right)^{\frac{1}{x}} = 1^\infty = f.i. = \lim_{x\to 0^-}\left(1+\frac{3^x-1}{2}\right)^{\frac{1}{x}} =$$

$$= \lim_{x\to 0^-}\left[\left(1+\frac{3^x-1}{2}\right)^{\frac{2}{3^x-1}}\right]^{\frac{1}{2}\frac{3^x-1}{x}} = e^{\frac{1}{2}\ln 3} = e^{\ln\sqrt{3}} = \sqrt{3}$$

$$\begin{cases} \lim_{x\to 0^-}\left(1+\frac{3^x-1}{2}\right)^{\frac{2}{3^x-1}} = \lim_{y\to 0^-}(1+y)^{\frac{1}{y}} = e \qquad y=\frac{3^x-1}{2} \\ \lim_{x\to 0^-}\frac{3^x-1}{x} = \ln 3 \end{cases}$$

mentre

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \ln\frac{x^2+3x}{sen\,x^2} = \lim_{x\to 0^+} \ln \underbrace{\frac{x^2}{sen\,x^2}}_{\to 1}\cdot\underbrace{\frac{(x+3)}{x}}_{\to +\infty} = \lim_{y\to+\infty}\ln y = +\infty$$

f ha in $x=0$ discontinuità di II^a specie $\forall\,\alpha\in\mathbb{R}$

poiché il limite destro vale sempre $+\infty$.