Corso di laurea in Chimica
Esame di
ISTITUZIONI di MATEMATICHE I

18 luglio 2025

1. Risolvere la seguente equazione
$$iz^6 - 2z^3 + 3i = 0.$$

2. Determinare il dominio della funzione
$$y = \sqrt{\frac{\lg^2_{\frac{1}{3}} x + \lg_{\frac{1}{3}}(27x^2)}{2^{\sqrt{x}} - 2^{1-x}}}.$$

3. Calcolare i seguenti limiti:
$$\lim_{x\to 0}\left(\sqrt{x^2+x} - x\right)\sin\frac{1}{x}, \qquad \lim_{x\to\pm\infty}\left(\sqrt{x^2+x} - x\right)\sin\frac{1}{x}.$$

4. Studiare e disegnare il grafico approssimativo della funzione
$$f(x) = \ln\sqrt{x^2+2x}.$$

5. Data la funzione
$$g(x) = x^3 e^{-x},$$

   a) calcolarne l'integrale indefinito;

   b) calcolare, se hanno senso, gli integrali di g su $[0,1]$, $[0,+\infty)$ e su $\mathbf{R}$.

Svolgimento

1. Posto $w = z^3$ l'equazione diventa $iw^2 - 2w + 3i = 0$ e quindi

$$w_{1/2} = \frac{1 \pm \sqrt{1-3i^2}}{i} = \frac{1 \pm \sqrt{4}}{i} = \begin{cases} \frac{3}{i} = -3i \\ -\frac{1}{i} = i \end{cases}$$

$$z^3 = -3i \Rightarrow z = \sqrt[3]{-3i} = \sqrt[3]{3\left(\cos\tfrac{3}{2}\pi + i \operatorname{sen}\tfrac{3}{2}\pi\right)} = \left\{\sqrt[3]{3}\left(\cos\frac{\frac{3}{2}\pi + 2k\pi}{3} + i \operatorname{sen}\frac{\frac{3}{2}\pi + 2k\pi}{3}\right) \Big| k \in \{0,1,2\}\right\}$$

da cui si ottengono

$$z_0 = \sqrt[3]{3}\left(\cos\tfrac{\pi}{2} + i \operatorname{sen}\tfrac{\pi}{2}\right) = \sqrt[3]{3}\, i$$

$$z_1 = \sqrt[3]{3}\left(\cos\tfrac{7}{6}\pi + i \operatorname{sen}\tfrac{7}{6}\pi\right) = \sqrt[3]{3}\left(-\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i\right)$$

$$z_2 = \sqrt[3]{3}\left(\cos\tfrac{11}{6}\pi + i \operatorname{sen}\tfrac{11}{6}\pi\right) = \sqrt[3]{3}\left(+\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i\right)$$

$$z^3 = i \Rightarrow z = \sqrt[3]{i} = \sqrt[3]{\cos\tfrac{\pi}{2} + i \operatorname{sen}\tfrac{\pi}{2}} = \left\{\cos\frac{\frac{\pi}{2} + 2k\pi}{3} + i \operatorname{sen}\frac{\frac{\pi}{2} + 2k\pi}{3} \Big| k \in \{0,1,2\}\right\}$$

da cui si ha

$$z_4 = \cos\tfrac{\pi}{6} + i \operatorname{sen}\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i$$

$$z_5 = \cos\tfrac{5}{6}\pi + i \operatorname{sen}\tfrac{5}{6}\pi = -\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i$$

$$z_6 = \cos\tfrac{3}{2}\pi + i \operatorname{sen}\tfrac{3}{2}\pi = -i$$

L'equazione ha 6 radici.

2. $$\begin{cases} \dfrac{\lg^2_{\frac{1}{3}} + \lg_{\frac{1}{3}}(27x^2)}{2^{\sqrt{x}} - 2^{1-x}} \geq 0 \\ x > 0 \end{cases}$$

Applico la regola del segno per $x > 0$

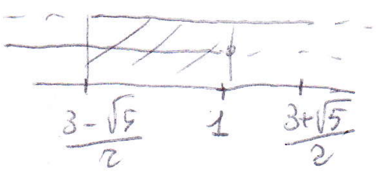
$N \geq 0$ $\quad \lg^2_{\frac{1}{3}} + \lg_{\frac{1}{3}} 27 + \lg_{\frac{1}{3}} x^2 \geq 0 \iff \lg^2_{\frac{1}{3}} + 2\lg_{\frac{1}{3}}|x| + \lg_{\frac{1}{3}}\left(\tfrac{1}{3}\right)^{-3} \geq 0$

$\iff \lg^2_{\frac{1}{3}} x + 2\lg_{\frac{1}{3}} x - 3 \geq 0$ (posso togliere il | | perché $x > 0$)

$(\lg_{\frac{1}{3}} x)_{1/2} = -1 \pm \sqrt{1+3} = \begin{cases} -3 \\ +1 \end{cases}$ $\quad \lg_{\frac{1}{3}} \leq -3 \vee \lg_{\frac{1}{3}} x \geq 1 \Rightarrow x \geq 27 \vee 0 < x \leq \frac{1}{3}$

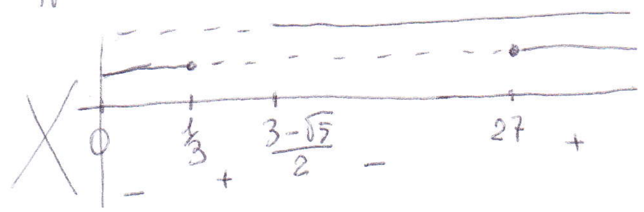
$D > 0$ $\quad 2^{\sqrt{x}} > 2^{1-x} \iff \sqrt{x} > 1-x \iff \begin{cases} x \geq 0 \\ 1-x < 0 \end{cases} \vee \begin{cases} 1-x \geq 0 \\ x > (1-x)^2 \end{cases} \iff$

$\begin{cases} x \geq 0 \\ x > 1 \end{cases} \vee \begin{cases} x \leq 1 \\ x^2 - 3x + 1 < 0 \end{cases} \iff x > 1 \vee \begin{cases} x \leq 1 \\ \frac{3-\sqrt{5}}{2} < x < \frac{3+\sqrt{5}}{2} \end{cases}$



$x > 1 \vee \frac{3-\sqrt{5}}{2} < x \leq 1 \iff x > \frac{3-\sqrt{5}}{2}$

Applicando la regola del segno per $x>0$ segue



$D=\left[\frac{1}{3},\frac{3-\sqrt{5}}{2}\right)\cup\left[27,+\infty\right)$

3. $\lim_{x\to 0}\underbrace{(\sqrt{x^2+x}-x)}_{\downarrow\, 0}\underbrace{\operatorname{sen}\frac{1}{x}}_{\downarrow\,\text{limitata}}=0$ perché prodotto di funzione infinitesima per funzione limitata.

$$\lim_{x\to+\infty}\underbrace{(\sqrt{x^2+x}-x)}_{+\infty-\infty}\operatorname{sen}\frac{1}{x}=\lim_{x\to+\infty}\frac{\cancel{x^2}+x-\cancel{x^2}}{\sqrt{x^2+x}+x}\cdot\operatorname{sen}\frac{1}{x}=\lim_{x\to+\infty}\frac{\cancel{x}}{2\cancel{x}}\operatorname{sen}\frac{1}{x}=0$$

$\sqrt{x^2+x}\sim\sqrt{x^2}=|x|=x$ se $x\to+\infty$

$$\lim_{x\to-\infty}\underbrace{(\sqrt{x^2+x}-x)}_{\downarrow\,+\infty}\operatorname{sen}\frac{1}{x}=\lim_{x\to-\infty}-2x\operatorname{sen}\frac{1}{x}=-2\lim_{x\to+\infty}\frac{\operatorname{sen}\frac{1}{x}}{\frac{1}{x}}=-2\lim_{y\to 0}\frac{\operatorname{sen}y}{y}=-2.$$

$\sqrt{x^2+x}\sim|x|=-x$ se $x\to-\infty$ $\qquad y=\frac{1}{x}$

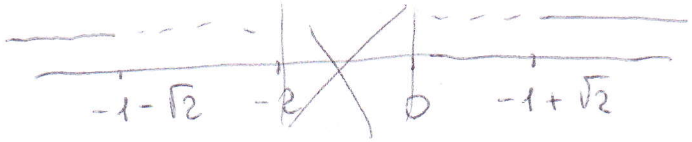
4. $f(x)=\ln\sqrt{x^2+2x}=\frac{1}{2}\ln(x^2+2x)$ $\qquad$ ID: $x^2+2x>0$ $\quad x<-2\vee x>0$

ID $=(-\infty,-2[\,\cup\,]0,+\infty)$

$\begin{cases} y=0\\ \ln(x^2+2x)=0\end{cases}\Rightarrow x^2+2x=1 \qquad x_{1,2}=-1\pm\sqrt{2}=\begin{cases}-1-\sqrt{2}<-2\\ -1+\sqrt{2}>0\end{cases}$

$(-1-\sqrt{2},0)$ $\quad(-1+\sqrt{2},0)$ intersezioni con asse x

$y>0\Rightarrow\ln(x^2+2x)>0\Rightarrow x^2+2x>1 \qquad x<-1-\sqrt{2}\vee x>-1+\sqrt{2}$



f positiva in $(-\infty,-1-\sqrt{2})\cup(-1+\sqrt{2},+\infty)$

f negativa in $(-1-\sqrt{2},-2)\cup(0,-1+\sqrt{2})$

Asintoti verticali

$\lim_{x\to-2^-}\frac{1}{2}\ln\underbrace{(x^2+2x)}_{y}=\frac{1}{2}\lim_{y\to0^+}\ln y=-\infty$ $\qquad x=-2$ As. vert. a dx

$\lim_{x\to0^+}\frac{1}{2}\ln(x^2+2x)=\frac{1}{2}\lim_{y\to0^+}\ln y=-\infty$ $\qquad x=0$ As. vert. a sin.

Asintoti orizzontali

$\lim_{x\to\pm\infty}\frac{1}{2}\ln(x^2+2x)=\frac{1}{2}\lim_{y\to+\infty}\ln y=+\infty$ $\qquad\nexists$ as. orizz.

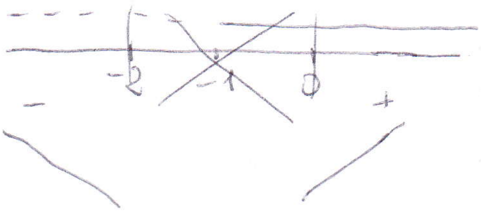
Poiché $\lim_{x\to\pm\infty}\frac{\frac{1}{2}\ln(x^2+2x)}{x}=\frac{1}{2}\lim_{x\to+\infty}\frac{\ln x^2}{x}=\frac{1}{2}\cdot2\lim_{x\to\pm\infty}\frac{\ln|x|}{x}=0$ per gli ordini,

$\nexists$ as. obliquo.

$$y' = \frac{1}{2}\,\frac{1}{x^2+2x}\,(2x+2) = \frac{x+1}{x^2+2x}$$

$ID(y') = ID(y)$ $\nexists$ pti di non derivabilità

$y' > 0 \iff x+1>0 \Rightarrow x > -1$



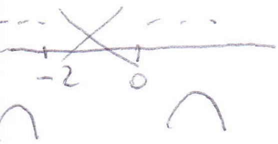
f decresce per $x < -2$

cresce per $x > 0$

$$y'' = \frac{x^2+2x-(x+1)(2x+2)}{(x^2+2x)^2} = \frac{x^2+2x-2(x+1)^2}{(x^2+2x)^2} = \frac{-x^2-2x-2}{(x^2+2x)^2}$$

$y'' > 0 \Rightarrow x^2+2x+2 < 0$

$\Delta < 0$ $\emptyset$



5. $$\int x^3 e^{-x}\,dx = \underset{P.P.}{\int x^3(-e^{-x})'\,dx =} x^3(-e^{-x}) - \int 3x^2(-e^{-x})\,dx = -x^3e^{-x} - 3\int x^2(e^{-x})'\,dx$$

$$\underset{P.P.}{=} -x^3e^{-x} - 3x^2e^{-x} + 6\int x e^{-x}\,dx \underset{PP}{=} -x^3e^{-x} - 3x^2e^{-x} - 6\left(x e^{-x} + \int e^{-x}\,dx\right)$$

$$= e^{-x}(-x^3 - 3x^2 - 6x - 6) + c$$

b) Poiché g è continua in $[0,1]$ chiuso e limitato, g è integ. nel senso classico in $[0,1]$ e $\int_0^1 g(x)\,dx = G(1) - G(0) = \frac{1}{e}(-1-3-6-6) - (-6) = -\frac{16}{e} + 6$.

Poiché g è continua in $\mathbb{R}$ e $\lim_{x\to+\infty} g(x) = 0$ per gli ordini mentre $\lim_{x\to-\infty} g(x) = -\infty$ segue che

$$\int_0^{+\infty} g(x)\,dx = \lim_{y\to+\infty}\int_0^y g(x)\,dx = \lim_{y\to+\infty}\,[\overset{\overset{0}{\uparrow}}{G(y)} - \overset{\overset{-6}{\shortparallel}}{G(0)}] = 6$$

$$\int_{-\infty}^{+\infty} g(x)\,dx = -\infty \quad \left(= \underbrace{\int_{-\infty}^0 g(x)\,dx}_{=-\infty} + \underbrace{\int_0^{+\infty} g(x)\,dx}_{\in\mathbb{R}}\right) = -\infty.$$