Corso di laurea in Chimica
Esame di
ISTITUZIONI di MATEMATICHE I

29 gennaio 2021

1. Studiare la seguente equazione nel campo complesso

$$z^2 + i \operatorname{Re} z = \frac{i}{8}\left(\frac{\sqrt{3} - 5 + i\left(\sqrt{3} - 1\right)}{2(1+i)} + \frac{3+4i}{1+3i}\right)^{33}$$

2. Calcolare

$$\lim_{x\to 0}\left(\frac{\ln(1+x)}{x} - e^{\frac{x}{2}}\right)^2 \frac{9}{e^{x^2} - 1 - \sin^2 x}.$$

3. Studiare insieme di definizione della funzione

$$y = \sqrt{\arctan\left(\frac{\ln^2 x^4 - 16\ln\sqrt{|x|} - 16\ln|e^2 x| - 8}{\ln|x|}\right)}.$$

4. Studiare la funzione

$$f(x) = x\sqrt{\left|1 + \frac{3}{x}\right|} + 1.$$

   e disegnarne il grafico approssimativo.

5. Data la funzione

$$g(x) = \frac{\sqrt{x+1} + 2}{(x+1)^2 - \sqrt{x+1}},$$

   i) calcolare l'integrale indefinito;

   ii) dire se g è integrabile in senso improprio in $\left]-1, -\frac{1}{2}\right]$, $]0,1]$ e in $[1, +\infty)$.

Svolgimento

1. Poiché

$$\frac{\sqrt{3}-5+i(\sqrt{3}-1)}{2(1+i)}\cdot\frac{1-i}{1-i}+\frac{3+4i}{1+3i}\cdot\frac{1-3i}{1-3i}=$$

$$\frac{\sqrt{3}-5+i(\sqrt{3}-1)+i(\sqrt{3}+5)+\sqrt{3}-1}{2\cdot 2}+\frac{3+4i-9i+12}{10}=$$

$$=\frac{2\sqrt{3}-6+2i}{4}+\frac{15-5i}{10}=\frac{\sqrt{3}-3+2i}{2}+\frac{3-i}{2}=\frac{\sqrt{3}}{2}+\frac{1}{2}i=\cos\frac{\pi}{6}+i\,\mathrm{sen}\frac{\pi}{6}$$

dalle formule di de Moivre si ha

$$\left(\cos\frac{\pi}{6}+i\,\mathrm{sen}\frac{\pi}{6}\right)^{33}=\cos\frac{33}{6}\pi+i\,\mathrm{sen}\frac{33}{6}\pi=\cos\left(\frac{3\pi}{2}+4\pi\right)+i\,\mathrm{sen}\left(\frac{3\pi}{2}+4\pi\right)=-i.$$

L'equazione diventa

$z^2+i\,\mathrm{Re}\,z=-\frac{i^2}{8}$ e posto $z=a+ib$ si ha

$$a^2-b^2+2abi+ai=\frac{1}{8}\iff\begin{cases}a^2-b^2=\frac{1}{8}\\ a(2b+1)=0\end{cases}\Rightarrow$$

$$\begin{cases}a=0\\ -b^2=\frac{1}{8}\end{cases}\ \emptyset\quad\vee\quad\begin{cases}b=-\frac{1}{2}\\ a^2=\frac{1}{4}+\frac{1}{8}=\frac{3}{8}\end{cases}\quad a=\pm\frac{1}{2}\sqrt{\frac{3}{2}}$$

L'equazione ha 2 sol

$$z_{1,2}=\pm\frac{1}{2}\sqrt{\frac{3}{2}}-\frac{1}{2}i$$

2. Applicando la formula di Taylor si ha:

$$\ln(1+x)=x-\frac{x^2}{2}+o(x^2)$$

$$e^{\frac{x}{2}}=1+\frac{x}{2}+\frac{1}{2}\left(\frac{x}{2}\right)^2+o(x^2)$$

$$e^{x^2}=1+x^2+\frac{x^4}{2}+o(x^4)$$

$$\mathrm{sen}^2x=\left(x-\frac{x^3}{6}+o(x^4)\right)^2=x^2-\frac{x^4}{3}+\frac{x^6}{36}+o(x^4)=x^2-\frac{x^4}{3}+o(x^4)$$

Sostituendo nel limite si ottiene

$$\lim_{x\to 0}\left(\frac{\ln(1+x)}{x}-e^{\frac{x}{2}}\right)^2\frac{9}{e^{x^2}-1-\mathrm{sen}^2x}=\lim_{x\to 0}\left(\frac{x-\frac{x^2}{2}+o(x^2)}{x}-1-\frac{x}{2}-\frac{x^2}{8}+o(x^2)\right)^2.$$

$$\cdot\frac{9}{\cancel{x^2}+\frac{x^4}{2}+o(x^4)-\cancel{x^2}+\frac{x^4}{3}+o(x^4)}=\lim_{x\to 0}\frac{9\left(\cancel{1}-\frac{x}{2}-\cancel{1}-\frac{x}{2}+o(x)\right)^2}{\frac{5}{6}x^4+o(x^4)}=\lim_{x\to 0}\frac{x^2}{x^4}=+\infty.$$

3. $\operatorname{arctg}\left(\dfrac{\ln^2 x^4 - 16\ln\sqrt{|x|} - 16\ln|e^2x| - 8}{\ln|x|}\right) \geq 0 \iff$

$$\frac{\ln^2 x^4 - 16\ln\sqrt{|x|} - 16\ln|e^2x| - 8}{\ln|x|} \geq 0$$

Poiché $\ln^2 x^4 = (\ln x^4)^2 = (4\ln|x|)^2 = 16\ln^2|x|$

$16\ln\sqrt{|x|} = 16\cdot\frac{1}{2}\ln|x| = 8\ln|x|$

$16\ln|e^2x| = 16(\ln e^2 + \ln|x|) = 16(2+\ln|x|)$

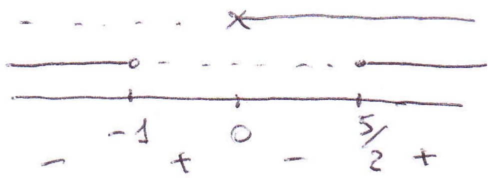
la disequazione diventa

$$\frac{\overset{2}{\cancel{16}}\ln^2|x| - \overset{1}{\cancel{8}}\ln|x| - \overset{4}{\cancel{32}} - \overset{2}{\cancel{16}}\ln|x| - \overset{1}{\cancel{8}}}{\ln|x|} \geq 0 \iff \frac{2\ln^2|x| - 3\ln|x| - 5}{\ln|x|} \geq 0$$

Applico regola del segno nel dominio della frazione: $\begin{cases}|x|>0\\ \ln|x|\neq 0\end{cases} \Rightarrow x \neq 0, \pm 1$.

Pongo $\ln|x| = t$ e studio

$\dfrac{2t^2-3t-5}{t} \geq 0 \Rightarrow$ $N \geq 0$: $t \leq -1 \vee t \geq \frac{5}{2}$; $D > 0$: $t > 0$.



$-1 \leq t < 0 \vee t \geq \frac{5}{2}$ cioè

$-1 \leq \ln|x| < 0 \vee \ln|x| \geq \frac{5}{2}$

$e^{-1} \leq |x| < 1 \vee |x| \geq e^{5/2} \iff -1 < x < -e^{-1} \vee e^{-1} < x < 1 \vee x \leq -e^{5/2} \vee x \geq e^{5/2}$

$ID = (-\infty, -e^{5/2}] \cup]-1, -e^{-1}] \cup [e^{-1}, 1[\cup [e^{5/2}, +\infty)$

4. ID: $\left|1+\frac{3}{x}\right| \geq 0 \quad \forall x \neq 0 \qquad ID = \mathbb{R}^*$ $\quad$ f non è pari né dispari

$\nexists$ intersezioni con asse y poiché $0 \notin ID$

$\begin{cases} y = 0 \\ x\sqrt{\left|1+\frac{3}{x}\right|} + 1 = 0 \end{cases} \Rightarrow -x\sqrt{\left|1+\frac{3}{x}\right|} = 1 \Rightarrow \begin{cases} -x \geq 0 \\ x^2\left(\sqrt{\left|1+\frac{3}{x}\right|}\right)^2 = 1 \end{cases}$

$\iff \begin{cases} x \leq 0 \\ 1+\frac{3}{x} \geq 0 \\ x^2+3x-1=0 \end{cases} \vee \begin{cases} x \leq 0 \\ 1+\frac{3}{x} < 0 \\ -x^2-3x-1=0 \end{cases} \iff$

$$\begin{cases} x \le 0 \\ x \le -3 \vee x > 0 \\ x_{1/2} = \dfrac{-3 \pm \sqrt{13}}{2} \end{cases} \vee \begin{cases} x \le 0 \\ -3 < x < 0 \\ x_{1/2} = \dfrac{-3 \pm \sqrt{5}}{2} \end{cases} \iff \begin{cases} x \le -3 \\ x_{1/2} = \begin{cases} \dfrac{-3-\sqrt{13}}{2} < -3 & \text{acc} \\ \dfrac{-3+\sqrt{13}}{2} > -3 & \text{non acc} \end{cases} \end{cases}$$

$$\vee \begin{cases} -3 < x < 0 \\ x = \begin{cases} \dfrac{-3-\sqrt{5}}{2} \in \left]-3,0\right[& \text{acc} \\ \dfrac{-3+\sqrt{5}}{2} \in \left]-3,0\right[& \text{acc.} \end{cases} \end{cases}$$

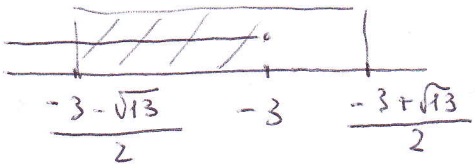
$\left(\dfrac{-3-\sqrt{13}}{2}, 0\right)$ $\left(\dfrac{-3-\sqrt{5}}{2}, 0\right)$ $\left(\dfrac{-3+\sqrt{5}}{2}, 0\right)$ intersezioni con asse x

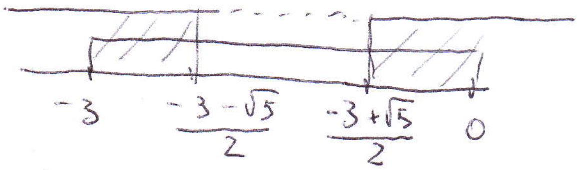
$$y > 0 \iff x\sqrt{\left|1+\frac{3}{x}\right|} + 1 > 0 \iff x > 0 \vee \begin{cases} x < 0 \\ 1 > -x\sqrt{\left|1+\frac{3}{x}\right|} \end{cases}$$

$$\iff x > 0 \vee \begin{cases} x < 0 \\ 1 > x^2\left|\frac{x+3}{x}\right| \end{cases} \iff \begin{cases} \frac{x+3}{x} \ge 0 \\ 1 > x^{\cancel{2}}\frac{x+3}{\cancel{x}} \end{cases} \vee \begin{cases} \frac{x+3}{x} < 0 \\ 1 > -x^{\cancel{2}}\frac{x+3}{\cancel{x}} \end{cases}$$

$$\iff x > 0 \vee \begin{cases} x < 0 \\ x \le -3 \vee x > 0 \\ x^2 + 3x - 1 < 0 \end{cases} \vee \begin{cases} x < 0 \\ -3 < x < 0 \\ x^2 + 3x + 1 > 0 \end{cases} \iff$$

$$x > 0 \vee \begin{cases} x \le -3 \\ \frac{-3-\sqrt{13}}{2} < x < \frac{-3+\sqrt{13}}{2} \end{cases} \vee \begin{cases} -3 < x < 0 \\ x < \frac{-3-\sqrt{5}}{2} \vee x > \frac{-3+\sqrt{5}}{2} \end{cases} \iff$$

$x > 0 \;\vee$ 

$\vee$ 

Conclusione

$$y > 0 \iff \frac{-3-\sqrt{13}}{2} < x < \frac{-3-\sqrt{5}}{2} \vee \frac{-3+\sqrt{5}}{2} < x < 0 \vee x > 0$$

Asintoti

$$\lim_{x \to 0} \underbrace{x}_{0 \text{ con ord } 1} \underbrace{\sqrt{\left|1+\frac{3}{x}\right|}}_{\to +\infty \text{ con ord } \frac{1}{2}} + 1 = 0 \cdot \infty + 1 = 0 + 1 \text{ (per gli ordini)} = 1 \quad \nexists \text{ A. V.}$$

f si può prolungare x continuità in $x = 0$ ponendole il valore $= 1$.

$$\lim_{x \to \pm\infty} x \cdot \sqrt{\left|1+\frac{3}{x}\right|} + 1 \quad \pm\infty \qquad \nexists \text{ as. orizz. a } \pm\infty$$

$$\lim_{x \to \pm\infty} \frac{x\sqrt{|1+3/x|} + 1}{x} = \lim_{x \to \pm\infty} \sqrt{\left|1+\frac{3}{x}\right|} + \frac{1}{x} = 1$$ (eventuale coeff. ang. A.ob.)

$$\lim_{x\to\pm\infty} x\sqrt{\left|1+\frac{3}{x}\right|}+1-x = \lim_{x\to\pm\infty} 3\,\frac{\sqrt{1+\frac{3}{x}}-1}{3\cdot\frac{1}{x}}+1 = \frac{3}{2}+1=\frac{5}{2} \quad (q)$$

tolgo $|\ |$ poiché $1+\frac{3}{x}>0 \iff x<-3 \vee x>0$

poiché $\lim_{y\to 0}\frac{(1+y)^{1/2}-1}{y}=\frac{1}{2}$

$y = x+\frac{5}{2}$ As. obliquo a $\pm\infty$

Studio della derivata prima

$$y' = \sqrt{\left|1+\frac{3}{x}\right|} + \cancel{x}\,\frac{1}{2\sqrt{\left|1+\frac{3}{x}\right|}}\,\frac{1+\frac{3}{x}}{\left|1+\frac{3}{x}\right|}\left(-\frac{3}{x^{\cancel{2}}}\right) = \sqrt{\left|1+\frac{3}{x}\right|} - \frac{3(x+3)}{2x^2\left|\frac{x+3}{x}\right|\sqrt{\left|1+\frac{3}{x}\right|}}$$

$$= \frac{2x^2\left|1+\frac{3}{x}\right|^2-3(x+3)}{2x^2\left|\frac{x+3}{x}\right|\sqrt{\left|1+\frac{3}{x}\right|}} = \frac{(x+3)(2x+6-3)}{2x^2\left|\frac{x+3}{x}\right|\sqrt{\left|1+\frac{3}{x}\right|}} = \frac{(x+3)(2x+3)}{2x^2\left|\frac{x+3}{x}\right|\sqrt{\left|1+\frac{3}{x}\right|}}$$

$ID(y')$: $\frac{x+3}{x}\neq 0 \Rightarrow x\neq 0 \wedge x\neq -3$ $\quad ID(y') = ID(y)-\{-3\}$

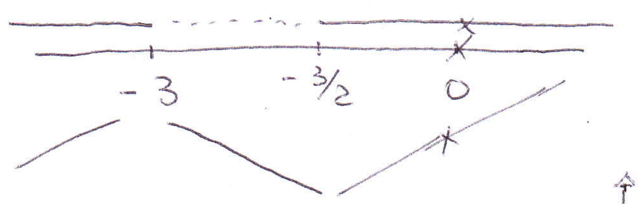
$$\lim_{x\to -3^-} y' = \lim_{x\to -3^-}\frac{(x+3)(2x+3)}{2x^2\,\frac{x+3}{x}\cdot\sqrt{1+\frac{3}{x}}} = \frac{-3}{2(-3)\cdot 0^+} = +\infty$$

$x=-3$ cuspide ascendente

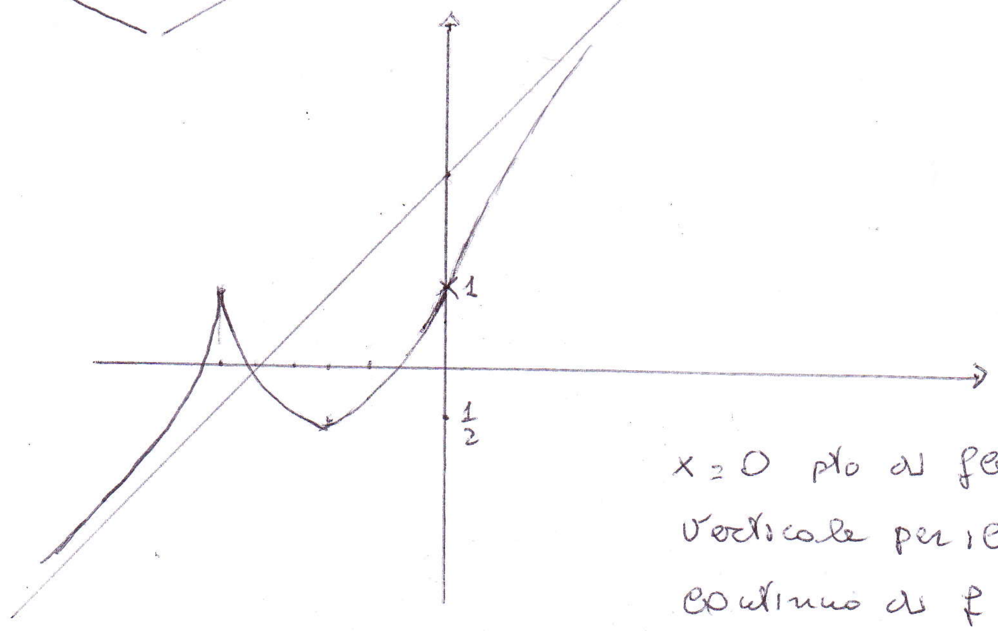
$$\lim_{x\to -3^+} y' = \lim_{x\to -3^+}\frac{(x+3)(2x+3)}{2x^2\left(-\frac{x+3}{x}\right)\sqrt{\left|1+\frac{3}{x}\right|}} = -\infty$$

$\lim_{x\to 0} y' = +\infty$ se prolunghiamo f per continuità in $x=0$, c'è pto a tg. verticale

$y'>0 \iff (x+3)(2x+3)>0 \iff x<-3 \vee x>-\frac{3}{2}$



$M\begin{cases} x=-3 \\ y=1\end{cases}$ $\quad m\begin{cases} x=-\frac{3}{2} \\ y=-\frac{3}{2}\sqrt{\left|1-3\cdot\frac{2}{3}\right|}+1=-\frac{1}{2}\end{cases}$



$x=0$ pto di flesso a tangente verticale per il prolungamento continuo di f

5\. i) $\displaystyle\int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}\,dx = \int \frac{t+2}{t^4-t}\cdot 2t\,dt$ posto $t=\sqrt{x+1}$

$t^2 = x+1 \quad dx = 2t\,dt$

$\displaystyle = 2\int \frac{t+2}{t^3-1}\,dt = 2\int \left(\frac{A}{t-1}+\frac{Bt+C}{t^2+t+1}\right)dt$ dove A, B, C tali che

$A(t^2+t+1)+(Bt+C)(t-1) = t+2 \iff \begin{cases} A+B=0 \\ A-B+C=1 \\ A-C=2 \end{cases} \quad \begin{cases} B=-A \\ A+A+A-2=1 \\ C=A-2 \end{cases}$

$\Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=-1 \end{cases}$ l'integrale diventa

$\displaystyle 2\int \frac{1}{t-1}\,dt - \int \frac{2t+1+1}{t^2+t+1}\,dt =$

$\displaystyle = 2\ln|t-1| - \ln(t^2+t+1) - \int \frac{4}{\underbrace{4t^2+4t+4}_{(2t+1)^2+3}}\,dx$

$\displaystyle = 2\ln\left|\frac{t-1}{t^2+t+1}\right| - \frac{2}{\sqrt{3}}\int \frac{2/\sqrt{3}}{\left(\frac{2t+1}{\sqrt{3}}\right)^2+1}\,dt = 2\ln\left|\frac{t-1}{t^2+t+1}\right| - \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2t+1}{\sqrt{3}} + c$

$\displaystyle = 2\ln\left|\frac{\sqrt{x+1}-1}{x+2+\sqrt{x+1}}\right| - \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt{x+1}+1}{\sqrt{3}} + c.$

ii) $I D(g)$: $\begin{cases} x+1\geq 0 \\ (x+1)^2 \neq \sqrt{x+1} \iff (x+1)^4 \neq x+1 \iff (x+1)[(x+1)^3-1]\neq 0 \end{cases}$ $\begin{matrix} x\neq -1 \\ x\neq 0 \end{matrix}$

$I D =]-1,0[\, \cup\,]0,+\infty)$

In $]-1,-\frac{1}{2}]$ g è continua e

$\displaystyle \lim_{x\to -1} g(x) \underset{y=x+1}{=} \lim_{y\to 0} \frac{y+2}{y^2-\sqrt{y}} = -\infty$ con ord $\frac{1}{2}<1 \Rightarrow g$ integ. in s. i.

In $]0,1]$ g è continua e

$\displaystyle \lim_{x\to 0} \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}} \underset{y=\sqrt{x+1}}{=} \lim_{y\to 1} \frac{y+2}{y^4-y} = \lim_{y\to 1} \frac{y+2}{y(y-1)(y^2+y+1)} = +\infty$ con ord 1

$\Rightarrow g$ non è integ. in s. i.

In $[1,+\infty)$ g è continua e

$\displaystyle \lim_{x\to +\infty} \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}} = \lim_{x\to+\infty} \frac{\sqrt{x}}{x^2} = 0$ con ord $\frac{3}{2}>1 \Rightarrow g$ integ. in s. i.