Corso di laurea in Chimica
Esame di
ISTITUZIONI di MATEMATICHE I

5 novembre 2020

1. Risolvere la seguente equazione nel piano complesso
$$\sqrt{2}z^4 - (1+i)\bar{z} = 0.$$

2. Calcolare il seguente limite
$$\lim_{x\to 0} \frac{\cos x^2 - e^{x^3} + x^3}{\arctan^4\left(\cos\sqrt{|x|} - 1\right)}.$$

3. Studiare la funzione
$$y = \left|2^{\frac{x-1}{x+2}} - 1\right|,$$
e tracciarne il grafico approssimativo.

4. Data la funzione
$$f(x) = \frac{4x-5}{2x-3+\sqrt{2x-1}},$$
i) determinare l'insieme delle primitive;

ii) stabilire, giustificando la risposta, se è integrabile in senso improprio in $\left[\frac{1}{2}, 1\right[$ e in $[3, +\infty[$.

Svolgimento

1. Posto $z = \rho(\cos\vartheta + i \,\mathrm{sen}\,\vartheta)$ si ha

$$\bar{z} = \rho(\cos\vartheta - i\,\mathrm{sen}\,\vartheta) = \rho(\cos(-\vartheta) + i\,\mathrm{sen}(-\vartheta)).$$

Poiché $z^4 = \rho^4(\cos 4\vartheta + i\,\mathrm{sen}\,4\vartheta)$ e $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\,\mathrm{sen}\frac{\pi}{4}\right)$

segue che

$$\cancel{\sqrt{2}}\rho^4(\cos 4\vartheta + i\,\mathrm{sen}\,4\vartheta) = \cancel{\sqrt{2}}\left(\cos\frac{\pi}{4} + i\,\mathrm{sen}\frac{\pi}{4}\right)\rho(\cos(-\vartheta) + i\,\mathrm{sen}(-\vartheta))$$

$$\Leftrightarrow \rho^4(\cos 4\vartheta + i\,\mathrm{sen}\,4\vartheta) = \rho\cos\left(\frac{\pi}{4} - \vartheta\right) + i\,\mathrm{sen}\left(\frac{\pi}{4} - \vartheta\right)$$

$$\Leftrightarrow \rho^4 = \rho \quad \wedge \quad \exists k \in \mathbb{Z} \ni' \ 4\vartheta = \frac{\pi}{4} - \vartheta + 2k\pi$$

$$\Leftrightarrow \rho(\rho^3 - 1) = 0 \quad \wedge \quad 5\vartheta = \frac{\pi}{4} + 2k\pi$$

$$\Leftrightarrow \rho = 0 \quad \vee \quad \begin{cases} \rho = 1 \\ \vartheta = \frac{\pi}{20} + \frac{2}{5}k\pi \end{cases} \qquad k \in \{0, 1, 2, 3, 4\}$$

In tutto l'eq. ha 6 soluzioni: $z = 0$ e $z_k = \cos\left(\frac{\pi}{20} + \frac{2}{5}k\pi\right) + i\,\mathrm{sen}(\ldots)$ con $k \in \{0, 1, 2, 3, 4, 5\}$.

2. Poiché $\cos x = 1 - \frac{x^2}{2} + o(x^3)$ $\quad e^x = 1 + x + \frac{x^2}{2} + o(x^2)$

segue che

$$\cos x^2 - e^{x^3} + x^3 = \cancel{1} - \frac{x^4}{2} + o(x^6) - \cancel{1} - \cancel{x^3} - \frac{x^6}{2} + o(x^6) + \cancel{x^3} \sim -\frac{x^4}{2}$$

mentre $\mathrm{arctg}^4(\cos\sqrt{|x|} - 1) \sim (\cos\sqrt{|x|} - 1)^4 \sim \left(-\frac{1}{2}(\sqrt{|x|})^2\right)^4 = \frac{1}{16}x^4$.

Quindi si conclude che

$$\lim_{x\to 0} \frac{\cos x^2 - e^{x^3} + x^3}{\mathrm{arctg}^4(\cos\sqrt{|x|} - 1)} = \lim_{x \to 0} \frac{-\frac{\cancel{x^4}}{2}}{\frac{1}{16}\cancel{x^4}} = -\frac{16}{2} = -8$$

3. $y = \left| 2^{\frac{x-1}{x+2}} - 1 \right|$ $\qquad$ ID: $x + 2 \neq 0$ $\quad \mathbb{R} - \{-2\}$

$y \geq 0 \quad \forall x \in ID$ $\qquad y = 0 \Leftrightarrow 2^{\frac{x-1}{x+2}} = 1 \Leftrightarrow \frac{x-1}{x+2} = 0 \Leftrightarrow x = 1$

$(1, 0)$ interes. con asse x $\qquad \begin{cases} x = 0 \\ y = |2^{-\frac{1}{2}} - 1| = 1 - \frac{1}{\sqrt{2}} \end{cases}$ $\left(0, 1 - \frac{1}{\sqrt{2}}\right)$ interes. asse y

$\lim_{x\to\pm\infty} \left|2^{\frac{x-1}{x+2}} - 1\right| = 2-1 = 1$ $\quad y = 1$ As. orizz. a $\pm\infty$

$\lim_{x\to -2^-} \left|2^{\frac{x-1}{x+2}} - 1\right| = \lim_{y\to+\infty} |2^y - 1| = +\infty$ poiché $\lim_{x\to -2^-} \frac{x-1}{x+2} = \frac{-3}{0^-} = +\infty$

$\lim_{x\to -2^+} \left|2^{\frac{x-1}{x+2}} - 1\right| = \lim_{y\to-\infty} |2^y - 1| = |0-1| = 1$ poiché $\lim_{x\to -2^+} \frac{x-1}{x+2} = -\infty$

Dunque $x = -2$ As. verticale solo per $x \to -2^-$

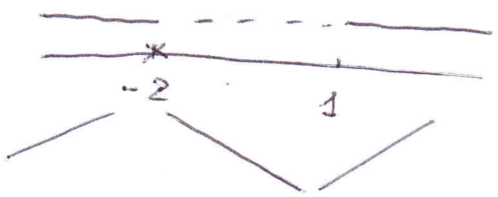
$$y' = \frac{2^{\frac{x-1}{x+2}} - 1}{\left|2^{\frac{x-1}{x+2}} - 1\right|}\, 2^{\frac{x-1}{x+2}} \lg 2 \,\frac{3}{(x+2)^2}$$

$\forall x \in ID$ t.c. $2^{\frac{x-1}{x+2}} - 1 \neq 0$

$\iff \forall x \neq -2,\ x \neq 1$

$y' \geq 0 \iff 2^{\frac{x-1}{x+2}} - 1 \geq 0$ (poiché $\lg 2 > 0$)

$\iff 2^{\frac{x-1}{x+2}} \geq 2^0 \iff \frac{x-1}{x+2} \geq 0 \iff x < -2 \vee x \geq 1$



m $\begin{cases} x = 1 \\ y = 0 \end{cases}$ f crescente in $(-\infty, -2[$ e in $]1, +\infty)$

f decrescente in $]-2, 1[$

Poiché
$$y' = \begin{cases} 2^{\frac{x-1}{x+2}} \cdot \frac{3\lg 2}{(x+2)^2} & \text{se } x < -2 \vee x > 1 \\ -\,2^{\frac{x-1}{x+2}} \cdot \frac{3\lg 2}{(x+2)^2} & \text{se } -2 < x < 1 \end{cases}$$

si ha che

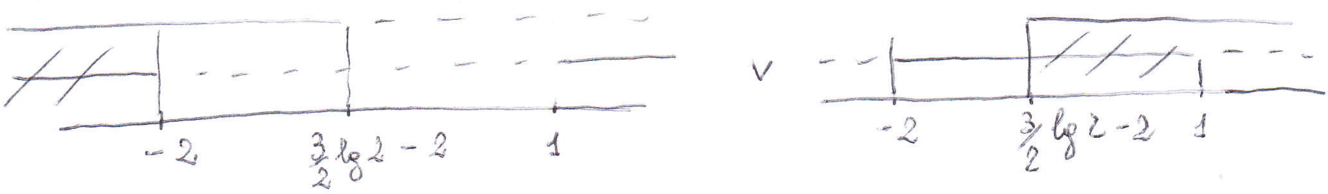
$$y'' = \begin{cases} 3\lg 2 \cdot \frac{2^{\frac{x-1}{x+2}}}{(x+2)^2}\left(\frac{3\lg 2}{(x+2)^2} - \frac{2}{x+2}\right) & \text{se } x < -2 \vee x > 1 \\ -3\lg 2\, \frac{2^{\frac{x-1}{x+2}}}{(x+2)^2}\left(\frac{3\lg 2}{(x+2)^2} - \frac{2}{x+2}\right) & \text{se } -2 < x < 1 \end{cases}$$

$y'' > 0 \iff \begin{cases} x < -2 \vee x > 1 \\ 3\lg 2 > 2(x+2) \end{cases} \vee \begin{cases} -2 < x < 1 \\ 3\lg 2 < 2(x+2) \end{cases}$

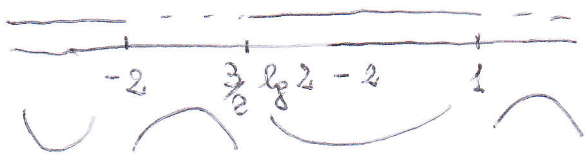
$x < \frac{3}{2}\lg 2 - \frac{4}{2}$

dove $\frac{3}{2}\lg 2 - 2 < 0$

$\iff \begin{cases} x < -2 \vee x > 1 \\ x < \frac{3}{2}\lg 2 - 2 \end{cases} \vee \begin{cases} -2 < x < 1 \\ x > -2 + \frac{3}{2}\lg 2 \end{cases}$



$y'' > 0 \iff x < -2 \ \vee \ \frac{3}{2}\lg 2 - 2 < x < 1$



f convesse in $(-\infty, -2[$ e in $]\frac{3}{2}\lg 2 - 2, 1[$

f concava in $]\frac{3}{2}\lg 2 - 2, 1[$

f ha 2 flessi di ascisse $x = 1$ e $x = \frac{3}{2}\lg 2 - 2$

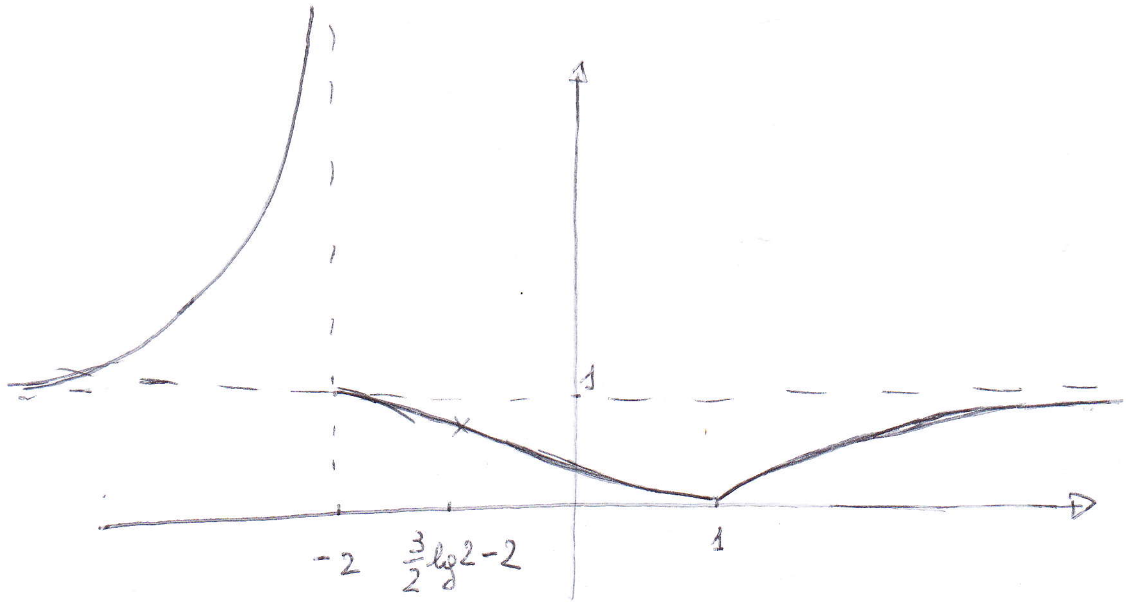
Si noti che $x = 1$ punto angoloso poiché si ha:

$$\lim_{x \to 1^-} y' = \lim_{x \to 1^-} -2^{\frac{x-1}{x+2}} \frac{3\lg 2}{(x+2)^2} = -\frac{3\lg 2}{9} \quad (= f'_-(1))$$

$$\lim_{x \to 1^+} y' = \lim_{x \to 1^+} 2^{\frac{x-1}{x+2}} \frac{3\lg 2}{(x+2)^2} = \frac{3\lg 2}{9} \quad (= f'_+(1))$$

Inoltre $\lim_{x \to -2^+} y' = \lim_{x \to -2^+} -\underbrace{2^{\frac{x-1}{x+2}}}_{\substack{\downarrow \\ 0 \text{ con ord esponenziale}}} \frac{3\lg 2}{\underbrace{(x+2)^2}_{\to +\infty \text{ con ord } 2}} = 0$ per gli ordini

dunque per $x \to -2^+$ $f(x) \to 1$ con pendenza $= 0$.



4. i) $\displaystyle\int \frac{4x-5}{2x-3+\sqrt{2x-1}}\,dx$ $\qquad t=\sqrt{2x-1} \Rightarrow t^2=2x-1$

$x=\dfrac{t^2+1}{2} \qquad dx = t\,dt$

$$= \int \frac{\cancel{4}^{\,2}\,\frac{t^2+1}{\cancel{2}}-5}{t^2+1-3+t}\,t\,dt = \int \frac{2t^3-3t}{t^2+t-2}\,dt$$

$$\begin{array}{rl|l} 2t^3 & -3t & t^2+t-2 \\ \hline 2t^3+2t^2-4t & & 2t-2 \\ \hline // \quad -2t^2+t & & \\ -2t^2-2t+4 & & \\ \hline // \quad 3t-4 & & \end{array}$$

$$= \int \left(2t-2+\frac{3t-4}{t^2+t-2}\right)dt \qquad t^2+t-2 = (t-1)(t+2)$$

$$= t^2-2t+\int\frac{3t-4}{(t-1)(t+2)}\,dt =$$

{ cerco A e B t.c. $\dfrac{3t-4}{(t-1)(t+2)} = \dfrac{A}{t-1}+\dfrac{B}{t+2} \iff \begin{cases}A+B=3\\ 2A-B=-4\end{cases}$ $\quad 3A \;//\; = -1$ $\qquad \begin{cases}A=-\frac{1}{3}\\ B=\frac{10}{3}\end{cases}$

$$= t^2-2t+\int\frac{-\frac{1}{3}}{t-1}\,dt+\frac{10}{3}\int\frac{1}{t+2}\,dt = t^2-2t-\frac{1}{3}\lg|t-1|+\frac{10}{3}\lg|t+2|+c$$

$$= 2x-1-2\sqrt{2x-1}-\frac{1}{3}\lg\left|\sqrt{2x-1}-1\right|+\frac{10}{3}\lg\left(\sqrt{2x-1}+2\right)+c.$$

ii) Prima di tutto studio il dominio di f.

$\begin{cases}2x-3+\sqrt{2x-1}\neq 0\\ 2x-1\geq 0\end{cases}$ $\qquad$ Studio l'eq. $2x-3+\sqrt{2x-1}=0$ e poi escludo i valori trovati

$$\iff \sqrt{2x-1} = 3-2x \iff \begin{cases}2x-1\geq 0\\ 3-2x\geq 0\\ 2x-1=(3-2x)^2\end{cases} \qquad \begin{cases}x\geq \frac{1}{2}\\ x\leq \frac{3}{2}\\ 4x^2-14x+10=0\end{cases}$$

L'eq. irrazionale ha unica sol $x=1$. $\qquad x_{1,2} = \begin{cases}1 & \text{acc}\\ \frac{5}{2} & \text{non acc}\end{cases}$

Segue che $ID(f): \begin{cases}x\geq\frac{1}{2}\\ x\neq 1\end{cases}$ cioè $ID(f)=[\frac{1}{2},1[\,\cup\,]3,+\infty)$

f è continua in $[\frac{1}{2},1[$ e in $[3,+\infty)$.

Poiché $\displaystyle\lim_{x\to 1^-}\frac{4x-5}{2x-3+\sqrt{2x-1}} = \frac{-1}{0} = \lim_{x\to 1^-}\frac{(4x-5)(2x-3-\sqrt{2x-1})}{(2x-3)^2-(\sqrt{2x-1})^2} =$

$$= \lim_{x\to 1^-}\frac{(4x-5)(2x-3-\sqrt{2x-1})}{4(x-1)(x-\frac{5}{2})} = \frac{(-1)(-2)}{0^+} = +\infty \text{ con ord } 1 \Rightarrow$$

f non è integ. in s.i. in $[\frac{1}{2},1[$.

Poiché $\displaystyle\lim_{x\to+\infty}\frac{4x-5}{2x-3+\sqrt{2x-1}} = 2\neq 0$ (trascuro infiniti di ordine inferiore) $\Rightarrow f$ non è integ. in s.i. in $[3,+\infty)$.