Corso di laurea in Chimica
Esame di
ISTITUZIONI di MATEMATICHE I

3 luglio 2018

1. Risolvere e disegnare nel piano complesso le soluzioni della seguente equazione
$$\overline{z-2i} = \frac{|z-1|^2}{6} - i^{21}.$$

2. Studiare la funzione
$$f(x) = \operatorname{arccotg}\left(e^{\frac{x^2}{1-x}} - 1\right)$$
e disegnarne il grafico approssimativo.

3. Determinare ordine e parte principale per $x \to +\infty$ di
$$g(x) = \left(\frac{\sqrt{x}}{\sqrt{x}+1}\right)^{\frac{1}{6\sqrt{x}}} - \sqrt{x}\sin\frac{1}{\sqrt{x}}.$$

4. Data la funzione
$$h(x) = \frac{1-\sin x+\cos x}{1+\sin x-\cos x}$$
i) calcolare l'integrale indefinito;

ii) dire, dopo averne determinato il dominio, se h è integrabile in $\left[0, \frac{\pi}{2}\right]$ e in $\left[\frac{\pi}{2}, \frac{3}{2}\pi\right[$.

5. Risolvere la disequazione
$$\left(\frac{1}{3}\right)^{\left|\sqrt{x^2-1}-x\right|} > \frac{1}{9}.$$

Svolgimento

1: Posto $z = x + iy$ si ha

$\overline{z-2i} = \overline{x+iy-2i} = \overline{x+i(y-2)} = x - i(y-2)$

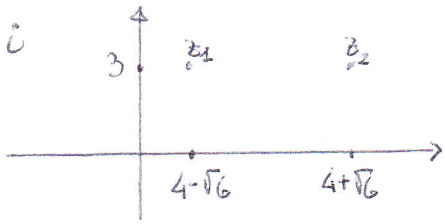
$|z-1|^2 = |x+iy-1|^2 = (x-1)^2 + y^2$ $\qquad i^{21} = i^{20} \cdot i = (i^4)^5 \cdot i = i$

e quindi l'equazione diventa

$x - i(y-2) = \dfrac{(x-1)^2 + y^2}{6} - i \iff \begin{cases} 6x = (x-1)^2 + y^2 \\ -y + 2 = -1 \end{cases} \iff$

$\begin{cases} y = 3 \\ 6x = x^2 - 2x + 1 + 9 \end{cases} \qquad x^2 - 8x + 10 = 0 \qquad x_{1,2} = 4 \pm \sqrt{6}$

L'equazione ha 2 soluzioni $z_{1,2} = 4 \pm \sqrt{6} + 3i$



2. $f(x) = \operatorname{arccotg}\left(e^{\frac{x^2}{1-x}} - 1\right)$

I.D.: $x \neq 1$ $\qquad \begin{cases} x = 0 \\ y = \operatorname{arccotg} 0 = \frac{\pi}{2} \end{cases}$ $\qquad y > 0 \quad \forall x \in I.D.$ (poiché la funzione arcotangente è sempre >0)

$\lim_{x \to -\infty} \operatorname{arccotg}\left(e^{\frac{x^2}{1-x}} - 1\right) = \operatorname{arccotg}(+\infty) = 0 \qquad y = 0$ A.O. sin.

$\lim_{x \to +\infty} \operatorname{arccotg}\left(e^{\frac{x^2}{1-x}} - 1\right) = \operatorname{arccotg}(-1) = \frac{3}{4}\pi \qquad y = \frac{3}{4}\pi$ A.O. dx

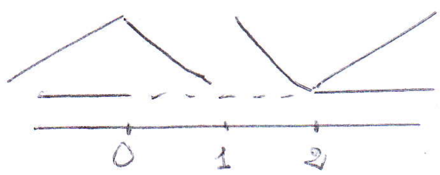
$\lim_{x \to +1^-} \operatorname{arccotg}\left(e^{\frac{x^2}{1-x}} - 1\right) = \operatorname{arccotg}(+\infty) = 0$

$\lim_{x \to 1^+} \operatorname{arccotg}\left(e^{\frac{x^2}{1-x}} - 1\right) = \operatorname{arccotg}(-1) = \frac{3}{4}\pi$

$y' = -\dfrac{1}{1 + \left(e^{\frac{x^2}{1-x}} - 1\right)^2} \cdot e^{\frac{x^2}{1-x}} \left(\dfrac{2x(1-x) + x^2}{(1-x)^2}\right) = \dfrac{e^{\frac{x^2}{1-x}}}{1 + \left(e^{\frac{x^2}{1-x}} - 1\right)^2} \dfrac{x^2 - 2x}{(1-x)^2} \quad \forall x \neq 1$

I.D.(y') = I.D.(y) $\quad \nexists$ pti angolosi o cuspidali

$y' > 0 \iff x^2 - 2x > 0 \qquad x < 0 \lor x > 2, \ x \neq 1$



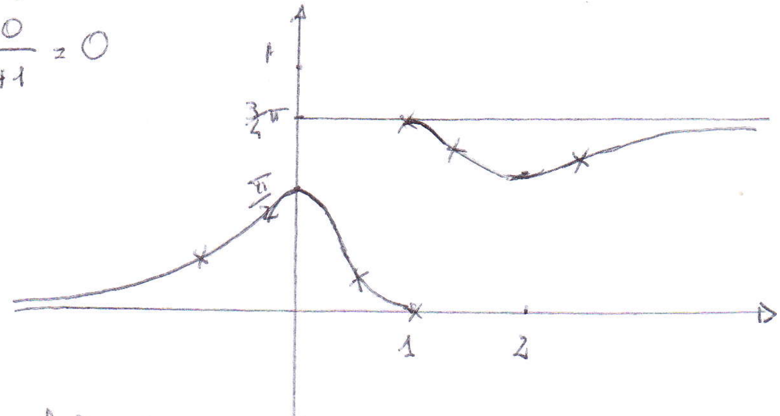
M $\begin{cases} x = 0 \\ y = \frac{\pi}{2} \end{cases}$ $\qquad$ m $\begin{cases} x = 2 \\ y = \operatorname{arccotg}(e^{-4} - 1) \end{cases}$

$$\lim_{x\to 1^-} y' = \lim_{x\to 1^-} \frac{e^{\frac{x^2}{1-x}}}{\underbrace{1+\left(e^{\frac{x^2}{1-x}}-1\right)^2}_{\downarrow}} \underbrace{\frac{x^2-2x}{(1-x)^2}}_{\to -\infty} = 0\cdot(-\infty) = 0$$ per gli ordini

$\frac{+\infty}{+\infty} \to 0$ per gli ordini essendo ord $e^{\frac{x^2}{1-x}} <$ ord $\left(e^{\frac{x^2}{1-x}}\right)^2$

$$\lim_{x\to 1^+} y' = \lim_{x\to 1^+} \frac{e^{\frac{x^2}{1-x}}}{\underbrace{1+\left(e^{\frac{x^2}{1-x}}-1\right)^2}_{\downarrow}} \frac{x^2-2x}{(1-x)^2} = \frac{0}{0} = 0$$ per gli ordini essendo ord $e^{\frac{x^2}{1-x}} >$ ord $(1-x)^2$

$\frac{0}{1+1} = 0$



Omettiamo lo studio del segno di y'', comunque si deduce che la funzione ha 4 flessi poiché per $x\to 1^-$ e $x\to 1^+$ $y'\to 0$.

3. $$\lim_{x\to+\infty} \underbrace{\left(\frac{\sqrt{x}}{\sqrt{x}+1}\right)^{\frac{1}{6\sqrt{x}}}}_{\downarrow\; 1^0=1} - \underbrace{\sqrt{x}\,\mathrm{sen}\,\frac{1}{\sqrt{x}}}_{\downarrow\; 1} = 0$$

$$\left(\frac{\sqrt{x}}{\sqrt{x}+1}\right)^{\frac{1}{6\sqrt{x}}} = e^{\frac{1}{6\sqrt{x}}\lg\left(\frac{\sqrt{x}}{\sqrt{x}+1}\right)} \to e^0 = 1$$

$$\sqrt{x}\,\mathrm{sen}\,\frac{1}{\sqrt{x}} = \frac{\mathrm{sen}\,\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}} \to 1$$

Per studiare ordine e parte principale, posto $\frac{1}{\sqrt{x}} = y$, per Taylor:

$$\left(\frac{1}{1+y}\right)^{\frac{y}{6}} - \frac{\mathrm{sen}\, y}{y} = e^{\frac{y}{6}\ln\left(\frac{1}{1+y}\right)} - \frac{\mathrm{sen}\, y}{y} = e^{-\frac{1}{6}y\ln(1+y)} - \frac{\mathrm{sen}\, y}{y}$$

$$\cong 1 - \frac{1}{6}y\ln(1+y) + \frac{1}{2}\frac{1}{36}y^2\ln^2(1+y) - \frac{y-\frac{y^3}{3!}+\frac{y^5}{5!}}{y}$$

$$\cong \cancel{1} - \frac{1}{6}y\left(y-\frac{y^2}{2}\right) - \frac{1}{72}y^2(y^2) - \cancel{1} + \frac{1}{6}y^2 - \frac{1}{120}y^4$$

$$= -\cancel{\frac{1}{6}y^2} + \frac{1}{12}y^3 - \frac{1}{72}y^4 + \cancel{\frac{1}{6}y^2} - \frac{1}{120}y^4 \sim \frac{1}{12}y^3 = \frac{1}{12}\frac{1}{x^{3/2}}$$

Dunque $g(x)\to 0$ per $x\to+\infty$ con ordine $3/2$ e parte principale $\frac{1}{12}x^{-3/2}$.

i) Poiché $\dfrac{1-\operatorname{sen}x+\cos x}{1+\operatorname{sen}x-\cos x} = \dfrac{-(1+\operatorname{sen}x+\cos x)+2}{1+\operatorname{sen}x-\cos x} = -1+\dfrac{2}{1+\operatorname{sen}x-\cos x}$

$$\int h(x)\,dx = \int\left(-1+\frac{2}{1+\operatorname{sen}x-\cos x}\right)dx = -x+2\int\frac{1}{1+\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}-\frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}}\,dx.$$

Posto $t=\operatorname{tg}\frac{x}{2} \Rightarrow x = 2\operatorname{arctg} t \Rightarrow dx = \frac{2}{1+t^2}dt$ il 2° integrale diventa

$$\int\frac{\cancel{1+t^2}}{\cancel{1}+t^2+2t-\cancel{1}+t^2}\cdot\frac{2}{\cancel{1+t^2}}\,dt = \int\frac{1}{t(t+1)}dt = \int\left(\frac{A}{t}+\frac{B}{t+1}\right)dt =$$

$$\int\frac{(A+B)t+A}{t(t+1)}dt \quad \left(\text{con } \begin{cases}A+B=0\\ A=1\end{cases} \Rightarrow \begin{cases}A=1\\ B=-1\end{cases} \text{ e quindi}\right)$$

$$=\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt = \lg|t|-\lg|t+1|+c = \lg\left|\frac{t}{t+1}\right|+c.$$

Dunque $\displaystyle\int h(x)\,dx = -x+2\lg\left|\frac{\operatorname{tg}\frac{x}{2}}{\operatorname{tg}\frac{x}{2}+1}\right|+c.$

ii) ID (h): $1+\operatorname{sen}x-\cos x \neq 0 \qquad \operatorname{sen}x - \operatorname{tg}\frac{\pi}{4}\cos x \neq -1$

$\operatorname{sen}x\cos\frac{\pi}{4}-\cos x\operatorname{sen}\frac{\pi}{4} \neq -\cos\frac{\pi}{4} \Leftrightarrow \operatorname{sen}\left(x-\frac{\pi}{4}\right)\neq\operatorname{sen}\left(-\frac{\pi}{4}\right) \Leftrightarrow$

$x-\cancel{\frac{\pi}{4}} \neq -\cancel{\frac{\pi}{4}}+2k\pi \ \vee\ x-\frac{\pi}{4}\neq\frac{5}{4}\pi+2k\pi \qquad \neq \operatorname{sen}\left(\frac{5}{4}\pi\right)$

$$ID = \mathbb{R}-\bigcup_{k\in\mathbb{Z}}\left\{2k\pi,\ \tfrac{3}{2}\pi+2k\pi\right\}$$

Poiché h è continua in $]0,\frac{\pi}{2}]$, calcolo

$$\lim_{x\to0^+}\frac{1-\operatorname{sen}x+\cos x}{1+\operatorname{sen}x-\cos x} = \frac{2}{0^+} = +\infty \quad \text{con ordine 1 poiché}$$

$$h(x)\sim\frac{1-x+1-\frac{x^2}{2}}{\cancel{1}+x-\cancel{1}+\frac{1}{2}x^2} \simeq \frac{2}{x}$$ Dunque h non è integrabile in s.i. in $[0,\frac{\pi}{2}]$.

Poiché

$$\lim_{x\to\frac{3}{2}\pi^-}\frac{1-\operatorname{sen}x+\cos x}{1+\operatorname{sen}x-\cos x} = \frac{2}{0} = \infty \quad \text{con ordine 1 essendo}$$

$$\lim_{x\to\frac{3}{2}\pi^-}h(x) = \lim_{\substack{y\to0\\ x-\frac{3}{2}\pi=y}}\frac{1-\operatorname{sen}\left(y+\frac{3}{2}\pi\right)+\cos\left(y+\frac{3}{2}\pi\right)}{1+\operatorname{sen}\left(y+\frac{3}{2}\pi\right)-\cos\left(y+\frac{3}{2}\pi\right)} =$$

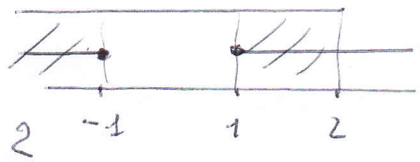
$$= \lim_{y\to 0^-} \frac{1+\cos y+\operatorname{sen} y}{1-\cos y+\operatorname{sen} y} = \lim_{y\to 0^-} \frac{2}{\frac{1}{2}y^2+y} = +\infty \quad \text{con ord. } 1$$

segue che f non è integ. in s.g. in $[\frac{\pi}{2}, \frac{3}{2}\pi[$.

5. $$\left(\frac{1}{3}\right)^{|\sqrt{x^2-1}-x|} > \left(\frac{1}{3}\right)^2 \iff |\sqrt{x^2-1}-x| < 2 \iff$$

$$-2 < \sqrt{x^2-1} - x < 2 \iff \begin{cases} \sqrt{x^2-1} > x-2 & (a) \\ \sqrt{x^2-1} < x+2 & (b) \end{cases}$$

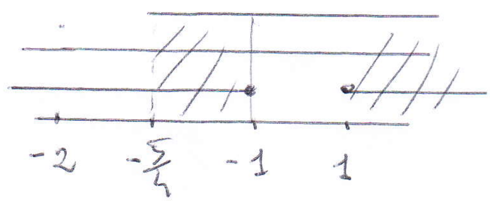
a) $\iff \begin{cases} x^2-1 \geq 0 \\ x-2<0 \end{cases} \vee \begin{cases} x-2 \geq 0 \\ \cancel{x^2}-1 > \cancel{x^2}-4x+4 \end{cases} \iff \begin{cases} x \leq -1 \vee x \geq 1 \\ x<2 \end{cases} \vee$



$\begin{cases} x \geq 2 \\ x > \frac{5}{4} \end{cases} \iff x \leq -1 \vee 1 \leq x < 2 \vee x \geq 2$

$\iff x \leq -1 \vee x \geq 1$

b) $\begin{cases} x^2-1 \geq 0 \\ x+2 \geq 0 \\ \cancel{x^2}-1 < \cancel{x^2}+4x+4 \end{cases} \iff \begin{cases} x \leq -1 \vee x \geq 1 \\ x \geq -2 \\ x > -\frac{5}{4} \end{cases}$



$-\frac{5}{4} < x \leq 1 \vee x \geq 1$

La diseq è soddisfatta per $\begin{cases} x \leq -1 \vee x \geq 1 \\ -\frac{5}{4} < x \leq 1 \vee x \geq 1 \end{cases}$ cioè

$-\frac{5}{4} < x \leq 1 \vee x \geq 1$