Corso di laurea in Chimica
Esame di
ISTITUZIONI di MATEMATICHE I

5 aprile 2018

1. Risolvere la seguente equazione nel campo complesso
$$|z|\,(3\,|z|-2)=z^3.$$

2. Studiare la funzione
$$f(x)=\ln\left|1+\frac{1}{x+1}\right|-\frac{1}{x+2}$$
e disegnarne il grafico approssimativo.

3. Calcolare il seguente limite
$$\lim_{x\to 0}\frac{2\ln(1+\sin x)-2\sin x+x^2}{[\sin(\pi\cos x)-\sin(\pi e^{-x})]\,\sqrt{1+x^3}}.$$

4. Dire per quali valori di $a\in\mathbf{R}$ è finito
$$\int_a^{+\infty}\frac{1}{\sqrt{x-2}(1-\sqrt[3]{(x-2)^2}}dx.$$
Calcolarlo per $a=4$. (Se si è in difficoltà, calcolare almeno l'integrale indefinito).

5. Risolvere la disequazione
$$\ln\left(e^{\frac{1}{x}}+e^{\frac{1}{2x}}-1\right)<\frac{3}{2x}.$$

Svolgimento

1. Posto $z = \rho(\cos\theta + i \operatorname{sen}\theta)$ si ha

$$\rho(3\rho-2) = \rho^3(\cos 3\theta + i \operatorname{sen} 3\theta) \iff \begin{cases} \rho(3\rho-2) = \rho^3 \cos 3\theta \\ \rho^3 \operatorname{sen} 3\theta = 0 \end{cases} \iff$$

$$\begin{cases} \rho = 0 \\ 0=0 \end{cases} \vee \begin{cases} \operatorname{sen} 3\theta = 0 \\ \rho(3\rho-2) = \rho^3 \cos 3\theta \end{cases} \iff z=0 \vee \begin{cases} 3\theta = 2k\pi \\ \rho(3\rho-2) = \rho^3 \end{cases} \vee \begin{cases} 3\theta' = \pi + 2k\pi \\ \rho(3\rho-2) = -\rho^3 \end{cases}$$

$$\iff \begin{cases} \theta_k = \frac{2}{3}k\pi \\ \rho^3 - 3\rho^2 + 2\rho = 0 \end{cases} \vee \begin{cases} \theta'_k = \frac{\pi}{3} + \frac{2}{3}k\pi \\ \rho^3 + 3\rho^2 - 2\rho = 0 \end{cases} \vee z = 0 \iff \begin{cases} \theta_k = \frac{2}{3}k\pi \\ \rho_{1,2} = \frac{3 \pm 1}{2} = \left\langle \begin{matrix} 1 \\ 2 \end{matrix} \right. \end{cases} \vee z = 0$$

$$\vee \begin{cases} \theta'_k = \frac{\pi}{3} + \frac{2}{3}k\pi \\ \rho_{1,2} = \frac{-3 \pm \sqrt{17}}{2} = \left\langle \begin{matrix} \frac{-3-\sqrt{17}}{2} < 0 \quad \text{Non acc.} \\ \frac{-3+\sqrt{17}}{2} > 0 \quad \text{acc.} \end{matrix} \right. \end{cases}$$

L'equazione ha le seguenti sol.:

$z = 0$

$z = \cos \frac{2}{3}\pi + i \operatorname{sen} \frac{2}{3}\pi = -\frac{\sqrt{3}}{2} + \frac{1}{2} i$

$z = \cos \frac{4}{3}\pi + i \operatorname{sen} \frac{4}{3}\pi = -\frac{\sqrt{3}}{2} - \frac{1}{2} i$

$z = \cos 2\pi + i \operatorname{sen} 2\pi = 1$

$z = 2(\cos \frac{2}{3}\pi + i \operatorname{sen} \frac{2}{3}\pi) = -\sqrt{3} + i$

$z = 2(\cos \frac{4}{3}\pi + i \operatorname{sen} \frac{4}{3}\pi) = -\sqrt{3} - i$

$z = 2(\cos 2\pi + i \operatorname{sen} 2\pi) = 2$

$z = \frac{-3+\sqrt{17}}{2}(\cos \frac{\pi}{3} + i \operatorname{sen} \frac{\pi}{3}) = \frac{-3+\sqrt{17}}{4}(\sqrt{3}+i)$

$z = \frac{-3+\sqrt{17}}{2}(\cos \pi + i \operatorname{sen} \pi) = \frac{3-\sqrt{17}}{2}$

$z = \frac{-3+\sqrt{17}}{2}(\cos \frac{5}{3}\pi + \operatorname{sen} \frac{5}{3}\pi) = -\frac{3+\sqrt{17}}{4}(\sqrt{3}-i)$

Dunque ci sono in tutto 10 soluzioni.

2. ID: $\begin{cases} \left|1 + \frac{1}{x+1}\right| > 0 \\ x + 2 \neq 0 \end{cases} \iff \begin{cases} \frac{x+2}{x+1} \neq 0 \\ x \neq -2 \end{cases} \iff \begin{cases} x \neq -1 \\ x \neq -2 \end{cases}$ $\quad ID = \mathbb{R} - \{-1, -2\}$

$\begin{cases} x = 0 \\ y = \ln 2 - \frac{1}{2} \end{cases}$ Non studio $y = 0$ né $y > 0$ (servirebbe metodo grafico)

$\lim_{x \to \pm\infty} \ln\left|1 + \frac{1}{x+1}\right| - \frac{1}{x+2} = \ln 1 = 0$ $\quad y = 0$ As. orizzontale a $\pm\infty$

$\lim_{x \to -2^+} \ln\left|1 + \frac{1}{x+1}\right| - \frac{1}{x+2} = -\infty - \infty = \infty$

$x = -2$ A.V. completo

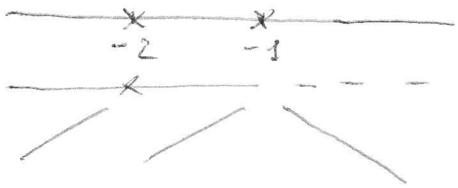
$\lim_{x \to -2^-} \ln\left|1 + \frac{1}{x+1}\right| - \frac{1}{x+2} = -\infty + \infty = +\infty$ per gli ordini

$$\lim_{x\to -1^{\mp}} \ln\left|1+\frac{1}{x+1}\right| - \frac{1}{x+2} = +\infty \qquad x=-1 \quad \text{A. V. completo}$$

$$y' = \frac{1}{1+\frac{1}{x+1}}\cdot\left(-\frac{1}{(x+1)^2}\right)+\frac{1}{(x+2)^2} = \frac{\cancel{x+1}}{x+2}\cdot\left(-\frac{1}{(x+1)^{\cancel{2}}}\right)+\frac{1}{(x+2)^2} = -\frac{1}{(x+2)(x+1)}+\frac{1}{(x+2)^2}$$

$$= \frac{-\cancel{x}-2+\cancel{x}+1}{(x+1)(x+2)^2} = -\frac{1}{(x+1)(x+2)^2} \qquad \mathrm{ID}(y') = \mathrm{ID}(y)$$

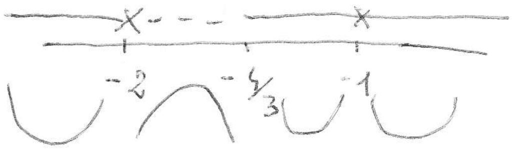
$y' > 0 \iff x+1<0 \iff x<-1$



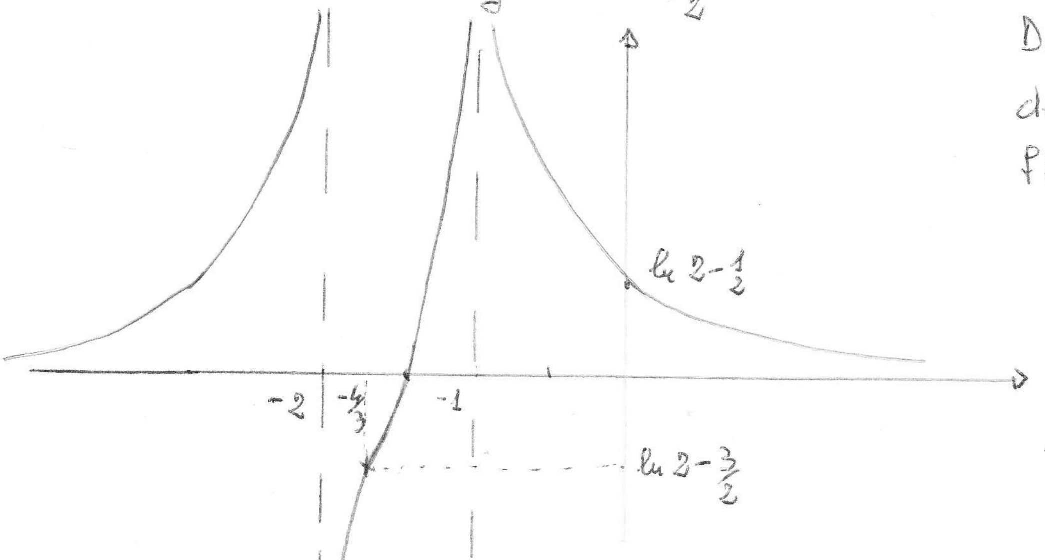
∄ estremi relativi di f

$$y'' = \frac{(x+2)^2+2(x+2)(x+1)}{(x+1)^2(x+2)^4} = \frac{(x+2)(3x+4)}{(x+1)^2(x+2)^4} \qquad y''>0 \iff (x+2)(3x+4)>0$$

$$\iff x<-2 \ \vee \ x>-\frac{4}{3}$$



$$F\begin{cases} x=-\frac{4}{3} \\ y=\ln 2-\frac{3}{2}<0 \end{cases}$$



Dal grafico si deduce che $f(x)=0 \iff x=\alpha$ con $-\frac{4}{3}<\alpha<-1$ mentre $f(x)>0 \iff$ $x<-2 \vee \alpha<x<-1 \vee x>-1$

Poiché $\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)$

$\text{sen}\, p - \text{sen}\, q = 2\cos\frac{p+q}{2}\,\text{sen}\,\frac{p-q}{2}$ (formule di prostaferesi)

$$\lim_{x\to 0}\frac{2\ln(1+\text{sen}\,x)-2\,\text{sen}\,x+x^2}{\text{sen}(\pi\cos x)-\text{sen}(\pi e^{-x})}\cdot\underbrace{\frac{1}{\sqrt{1+x^3}}}_{\downarrow\,1} = \lim\frac{2\left(\cancel{\text{sen}\,x}-\frac{\text{sen}^2x}{2}+\frac{\text{sen}^3x}{3}\right)-2\,\cancel{\text{sen}\,x}+x^2}{\underbrace{2\cos\frac{\pi}{2}(\cos x+e^{-x})}_{\downarrow\,-2}\,\underbrace{\text{sen}\,\frac{\pi}{2}(\cos x-e^{-x})}_{\sim\frac{\pi}{2}(\cos x-e^{-x})}}$$

$$=\lim_{x\to 0}\frac{\overbrace{x^2-\text{sen}^2x}^{\geq 2}+\overbrace{\tfrac{2}{3}\text{sen}^3x}^{3}}{-2\cdot\frac{\pi}{2}\underbrace{\left(\cancel{1}-\frac{x^2}{2}-\cancel{1}+x\right)}_{1}} = 0$$

per gli ordini poiché ord(Num) ≥ 2 mentre ord(Denom) $= 1$

N.B. Si poteva anche evitare prostaferesi e, dopo aver trasformato il limite di partenza opportunamente, applicare Teor. di de l'Hôpital.

$$\lim_{x\to 0}\frac{x^2-\operatorname{sen}^2 x+\operatorname{sen}^3 x}{\operatorname{sen}(\pi\cos x)-\operatorname{sen}(\pi e^{-x})} = \lim_{x\to 0}\frac{x^3}{\operatorname{sen}(\pi\cos x)-\operatorname{sen}(\pi e^{-x})} \overset{H}{=}$$

poiché $x^2-\operatorname{sen}^2 x=(x-\operatorname{sen} x)(x+\operatorname{sen} x)\sim\frac{1}{3}x^3\cdot 2x=\frac{2}{3}x^4$ ha ord 4 e si trascura rispetto a $\operatorname{sen}^3 x\sim x^3$

$$=\lim_{x\to 0}\frac{3x^2}{\underbrace{\cos(\pi\cos x)(-\pi\operatorname{sen} x)}_{\to 0}+\underbrace{\cos(\pi e^{-x})\cdot e^{-x}}_{\to -1}}=\frac{0}{-1}=0.$$

4. ID: $\begin{cases} x-2>0 \\ 1-\sqrt[3]{(x-2)^2}\neq 0\end{cases} \iff \begin{cases} x>2 \\ \sqrt[3]{(x-2)^2}\neq 1\end{cases} \iff \begin{cases} x>2 \\ x-2\neq\pm 1 \Rightarrow x\neq 3,1\end{cases} \iff ID =]2,3[\,\cup\,]3,+\infty)$

Poiché
- $\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\frac{1}{\sqrt{x-2}}\cdot\frac{1}{1-\sqrt[3]{(x-2)^2}}=+\infty$ con ord $\frac{1}{2}<1\Rightarrow$ f integrabile in s.i. per $x\to 2$
- $\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{1}{\sqrt{x-2}}\cdot\frac{1}{1-\sqrt[3]{(x-2)^2}}=+\infty$ con ord $1\Rightarrow$ f non è integ. in s.i. per $x\to 2$

$\left(\lim_{x\to 3}\frac{1-\sqrt[3]{(x-2)^2}}{x-3}\overset{H}{=}\lim_{x\to 3}-\frac{2}{3}(x-2)^{-\frac{1}{3}}=-\frac{2}{3}\Rightarrow 1-\sqrt[3]{(x-2)^2}\to 0\right.$ con ord 1 per $\left.x\to 3\right)$

- $\lim_{x\to+\infty} f(x)=0$ con ord $\frac{1}{2}+\frac{2}{3}=\frac{7}{6}>1\Rightarrow$ f integ. per $x\to+\infty$

segue che $\int_a^{+\infty} f(x)\,dx$ è finito $\forall\, a>3$. Calcolo ora una primitiva di f.

$$\int\frac{1}{\sqrt{x-2}\left(1-\sqrt[3]{(x-2)^2}\right)}dx=\int\frac{6t^{\cancel{5}\,2}}{\cancel{t^3}(1-t^4)}dt=-6\int\frac{t^2}{t^4-1}dt \qquad x-2=t^6 \quad x=t^6+2 \quad dx=6t^5\,dt$$

$$\frac{t^2}{t^4-1}=\frac{t^2}{(t-1)(t+1)(t^2+1)}=\frac{A}{t-1}+\frac{B}{t+1}+\frac{Ct+D}{t^2+1}=\frac{A(t+1)(t^2+1)+B(t-1)(t^2+1)+(Ct+D)(t^2-1)}{t^4-1}$$

$$\Rightarrow\begin{cases}A+B+C=0\\ A-B+D=1\\ A+B-C=0\\ A-B-D=0\end{cases}$$

Sommo e sottraggo 1ª e 3ª e poi sommo e sottraggo 2ª e 4ª

$$\begin{cases} A+B=0\\ 2C=0 \\ A-B=1 \\ 2D=1\end{cases} \Rightarrow \begin{cases} C=0\\ D=\frac{1}{2} \\ A=-B \\ 2A=1\end{cases} \Rightarrow \quad A=\frac{1}{2} \qquad B=-\frac{1}{2} \qquad C=0 \qquad D=\frac{1}{2}$$

Dunque $\int f(x)\,dx = \int \frac{\frac{1}{2}}{t-1}\,dt - \int \frac{1/2}{t+1}\,dt + \int \frac{\frac{1}{2}}{t^2+1}\,dt = \frac{1}{2}\lg|t-1| - \frac{1}{2}\lg|t+1| + \frac{1}{2}\operatorname{arctg} t + c$

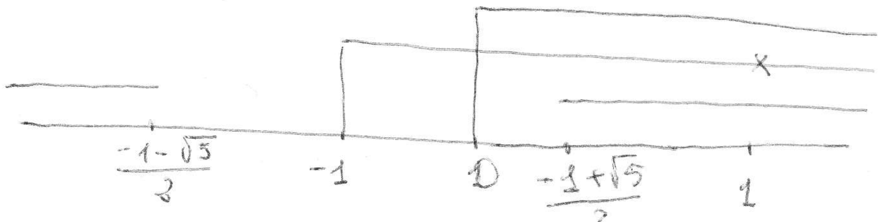
e quindi

$$\int_4^{+\infty} f(x)\,dx = \lim_{M\to+\infty} \int_4^M f(x)\,dx = \lim_{M\to+\infty}\left[\frac{1}{2}\lg\underbrace{\left|\frac{M-1}{M+1}\right|}_{\ln 1 = 0} + \underbrace{\frac{1}{2}\operatorname{arctg} M}_{\frac{1}{2}\cdot\frac{\pi}{2}} - \frac{1}{2}\lg\left|\frac{3}{5}\right| - \frac{1}{2}\operatorname{arctg} 4\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}\lg\frac{3}{5} - \frac{1}{2}\operatorname{arctg} 4.$$

5. $\ln\left(e^{\frac{1}{x}} + e^{\frac{1}{2x}} - 1\right) < \frac{3}{2x} \iff \begin{cases} e^{\frac{1}{x}} + e^{\frac{1}{2x}} - 1 > 0 \\ e^{\frac{1}{x}} + e^{\frac{1}{2x}} - 1 < e^{\frac{3}{2x}} \end{cases}$ Posto $e^{\frac{1}{2x}} = t$: $e^{\frac{1}{x}} = t^2$, $e^{\frac{3}{2x}} = t^3$

$$\begin{cases} t^2 + t - 1 > 0 \\ t^3 - t^2 - t + 1 > 0 \\ t > 0 \end{cases} \Rightarrow \begin{cases} t < \frac{-1-\sqrt{5}}{2} \ \vee \ t > \frac{-1+\sqrt{5}}{2} \\ t^2(t-1) - (t-1) > 0 \end{cases} \qquad (t-1)^2(t+1) > 0 \quad t > -1,\ t \neq 1$$



$t > \frac{-1+\sqrt{5}}{2}$, $t \neq 1$

$$\begin{cases} e^{\frac{1}{2x}} > \frac{-1+\sqrt{5}}{2} \\ e^{\frac{1}{2x}} \neq 1 \end{cases} \Rightarrow$$

$$\begin{cases} \frac{1}{2x} > \ln\frac{-1+\sqrt{5}}{2} \\ \frac{1}{2x} \neq 0 \end{cases} \iff \begin{cases} \dfrac{1 - 2x\ln\frac{-1+\sqrt{5}}{2}}{2x} > 0 \\ x \neq 0 \end{cases} \iff x < \frac{1}{2\ln\left(\frac{-1+\sqrt{5}}{2}\right)} \ \vee \ x > 0$$

(N. B. $\ln\frac{-1+\sqrt{5}}{2} < 0$!)