# Corso di laurea in Chimica
# Esame
# ISTITUZIONI di MATEMATICHE I

**13 ottobre 2017**

1. Studiare la seguente equazione nel campo complesso

$$z^6 - 4i = iz^6 + 4\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{24} + i^{31} - \left(\frac{1+i}{1-i}\right)^{27}.$$

2. Determinare il campo di esistenza della funzione

$$f(x) = \ln\sqrt{\log_{\frac{1}{3}}\frac{|x|-1}{|x+1|}}.$$

3. Studiare la funzione

$$g(x) = \sqrt{x^2-1} + \left|x + \frac{3}{2}\right|$$

e disegnarne il grafico approssimativo.

4. Calcolare il seguente limite

$$\lim_{x\to+\infty}\frac{e^{\frac{1}{\sqrt{x}}} - e^{-\frac{1}{\sqrt{x}}} - 2\arcsin\frac{1}{\sqrt{x}}}{\sin\frac{1}{x} - \frac{1}{x}}.$$

5. Data la funzione

$$h(x) = \frac{1+\tan^2 x}{\tan x + 2}\sqrt{\frac{1-\tan x}{1+\tan x}},$$

i) calcolare l'integrale indefinito;

ii) dire, giustificando le risposte, se è integrabile in senso improprio in $]-\frac{\pi}{4}, 0,]$, in $[0, \frac{\pi}{4}[$ e in $[2, +\infty)$.

# Svolgimento

1. Poiché

$$4\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{24} = 4\left(\cos\frac{\pi}{6}+i\,\text{sen}\,\frac{\pi}{6}\right)^{24} = 4\left(\cos 4\pi + i\,\text{sen}\,2\pi\right) = 4$$

$$i^{31} = i^{28}\cdot i^{3} = (i^{4})^{7}\cdot i^{3} = 1^{7}\cdot i^{3} = -i$$

$$\left(\frac{1+i}{1-i}\right)^{27} = \left[\frac{(1+i)^2}{1-i^2}\right]^{27} = \left(\frac{\cancel{1}+\cancel{i^2}+\cancel{2}i}{\cancel{2}}\right)^{27} = i^{24}\cdot i^{3} = -i$$

l'equazione diventa $z^6 - 4i = iz^6 + 4 - \cancel{i} + \cancel{i}$ cioè

$$z^6(1-i) = 4+4i \iff z^6 = 4\,\frac{1+i}{1-i} \iff z^6 = 4i \quad \text{cioè}$$

$$z = \sqrt[6]{i} = \sqrt[6]{\cos\frac{\pi}{2}+i\,\text{sen}\,\frac{\pi}{2}} = \left\{\cos\frac{\frac{\pi}{2}+2k\pi}{6} + i\,\text{sen}\,\frac{\frac{\pi}{2}+2k\pi}{6} \;\middle|\; k\in\{0,1,..5\}\right\}$$

$z_0 = \cos\frac{\pi}{12} + i\,\text{sen}\,\frac{\pi}{12}$

$z_1 = \cos\frac{\frac{\pi}{2}+2\pi}{6} + i\,\text{sen}\,\frac{\frac{\pi}{6}+2\pi}{6} = \cos\frac{5}{12}\pi + i\,\text{sen}\,\frac{5}{12}\pi$

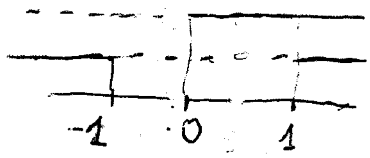
$z_2 = \cos\frac{\cancel{9}\,3}{\cancel{12}\,4}\pi + i\,\text{sen}\,\frac{\cancel{9}\,3}{\cancel{12}\,4}\pi$

$z_3 = \cos\frac{13}{12}\pi + i\,\text{sen}\,\frac{13}{12}\pi = -z_0$

$z_4 = -z_1 \qquad z_5 = -z_2$

2. $$\sqrt{\log_{\frac{1}{3}}\frac{|x|-1}{|x+1|}} > 0 \iff \log_{\frac{1}{3}}\frac{|x|-1}{|x+1|} > 0 \iff 0 < \frac{|x|-1}{|x+1|} < 1$$

$$\begin{cases} \dfrac{|x|-1}{|x+1|} > 0 \iff \begin{cases}|x|>1\\ x\neq -1\end{cases} \iff x<-1 \lor x>1 \\[2em] \dfrac{|x|-1}{|x+1|} < 1 \iff \begin{cases}|x|-1<|x+1|\\ x\neq -1\end{cases} \iff \begin{cases}x<-1\\ -x-1<-x-1\end{cases} \lor \begin{cases}-1<x\le 0\\ -x-1<+x+1\end{cases} \lor \begin{cases}x>0\\ x-1<x+1 \iff\end{cases} \end{cases}$$



$\emptyset \qquad -1<x\le 0 \;\lor\; x>0 \qquad x>-1$

$ID = ]+1,+\infty)$

3. $ID = (-\infty,-1]\cup[1,+\infty)$

$\begin{cases}x=1\\ y=0+\frac{5}{2}\end{cases} \quad \begin{cases}x=-1\\ y=\frac{1}{2}\end{cases} \quad \begin{cases}y=0\\ \emptyset\end{cases} \quad \begin{cases}y>0\\ \forall x\in ID\end{cases}$

$\lim_{x\to\pm\infty} g(x) = +\infty \qquad \nexists$ as. orizz. $\qquad \nexists$ as. verticali

$$\lim_{x\to+\infty} \frac{g(x)}{x} = \lim_{x\to+\infty} \frac{\sqrt{x^2-1}+x+\frac{3}{2}}{x} = \lim_{x\to+\infty} \frac{|x|\sqrt{1-\frac{1}{x^2}}+x+\frac{3}{2}}{x}$$

$$= \lim_{x\to+\infty} \cancel{x}\,\frac{\sqrt{1-\frac{1}{x^2}}+1+\frac{3}{2x}}{\cancel{x}} = 2 \quad \text{(eventuale coeff. ang.)}$$

$$\lim_{x\to+\infty} g(x)-2x = \lim_{x\to+\infty} \sqrt{x^2-1}+x+\frac{3}{2}-2x = \lim_{x\to+\infty}\left(\sqrt{x^2-1}+\frac{3}{2}-x\right)$$

$$= \lim_{x\to+\infty} \frac{x^2-1-\left(\frac{3}{2}-x\right)^2}{\sqrt{x^2-1}-\frac{3}{2}+x} = \lim_{x\to+\infty} \frac{\cancel{x}\left(3-\frac{13}{4x}\right)}{\cancel{x}\left(\sqrt{1-\frac{1}{x^2}}-\frac{3}{2x}+1\right)} = \frac{3}{2} \quad (q)$$

$y = 2x + \frac{3}{2}$ Asintoto obliquo a $+\infty$

$$\lim_{x\to-\infty} \frac{g(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{x^2-1}-x-\frac{3}{2}}{x} = \lim_{x\to-\infty} \frac{+\cancel{x}\left(-\sqrt{1-\frac{1}{x^2}}-1-\frac{3}{2x}\right)}{\cancel{x}} = -2$$

$$\lim_{x\to-\infty} g(x)+2x = \lim_{x\to-\infty} \sqrt{x^2-1}-x-\frac{3}{2}+2x = \lim_{x\to-\infty} \sqrt{x^2-1}+x-\frac{3}{2}$$

$$= \lim_{x\to-\infty} \frac{x^2-1-\left(x-\frac{3}{2}\right)^2}{\sqrt{x^2-1}-x+3/2} = \lim_{x\to-\infty} \frac{\cancel{x}\left(3-\frac{13}{4x}\right)}{\cancel{x}\left(-\sqrt{1-\frac{1}{x^2}}-1+\frac{3}{2x}\right)} = -\frac{3}{2}$$

$y = -2x - \frac{3}{2}$ Asintoto obliquo a $-\infty$

$$y' = \frac{x}{\sqrt{x^2-1}} + \frac{x+\frac{3}{2}}{\left|x+\frac{3}{2}\right|} = \begin{cases} \dfrac{x}{\sqrt{x^2-1}}+1 & x\in ID\cap\left]-\frac{3}{2},+\infty\right) = \left]-\frac{3}{2},-1\right[\cup\left]1,+\infty\right),\ x\neq\pm1 \\[2ex] \dfrac{x}{\sqrt{x^2-1}}-1 & x\in ID\cap\left(-\infty,-\frac{3}{2}\right) = \left(-\infty,-\frac{3}{2}\right[ \end{cases}$$

$$\lim_{x\to-\frac{3}{2}^+} y' = \lim_{x\to-\frac{3}{2}^+} \frac{x}{\sqrt{x^2-1}}+1 = \frac{-\frac{3}{2}}{\sqrt{\frac{9}{4}-1}}+1 = -\frac{3}{\sqrt{5}}+1 = f'_+\left(-\frac{3}{2}\right)$$

$$\lim_{x\to-\frac{3}{2}^-} y' = \lim_{x\to-\frac{3}{2}^-} \frac{x}{\sqrt{x^2-1}}-1 = -\frac{3}{\sqrt{5}}-1 = f'_-\left(-\frac{3}{2}\right)$$

$$\begin{cases} x = -\frac{3}{2} \\ y = \frac{\sqrt{5}}{2} \end{cases}$$ punto angoloso

$$\lim_{x\to1^+} y' = +\infty \qquad \lim_{x\to-1^-} y' = -\infty$$

$x = \pm 1$ pti di semicuspide

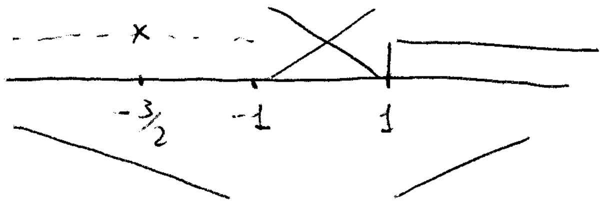
$$y'>0 \iff \begin{cases} x<-\frac{3}{2} \\ \frac{x}{\sqrt{x^2-1}}>1 \end{cases} \vee \begin{cases} -\frac{3}{2}<x<-1 \vee x>1 \\ \frac{x}{\sqrt{x^2-1}}>-1 \end{cases} \iff \begin{cases} -\frac{3}{2}<x<-1 \vee x>1 \\ -x<\sqrt{x^2-1} \end{cases}$$

$$\emptyset \qquad \iff \begin{cases} -\frac{3}{2}<x<-1 \vee x>1 \\ -x<0 \end{cases} \vee \begin{cases} -\frac{3}{2}<x<-1 \vee x>1 \\ -x>0 \\ \cancel{x^2}<\cancel{x^2}-1 \end{cases}$$

$$\emptyset$$

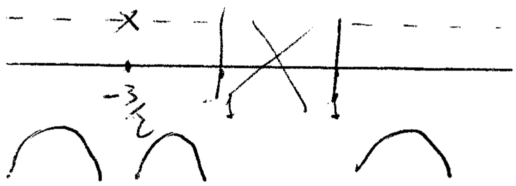
$$\iff \begin{cases} -\frac{3}{2}<x<-1 \vee x>1 \\ x>0 \end{cases}$$

$$y'>0 \iff x>1$$

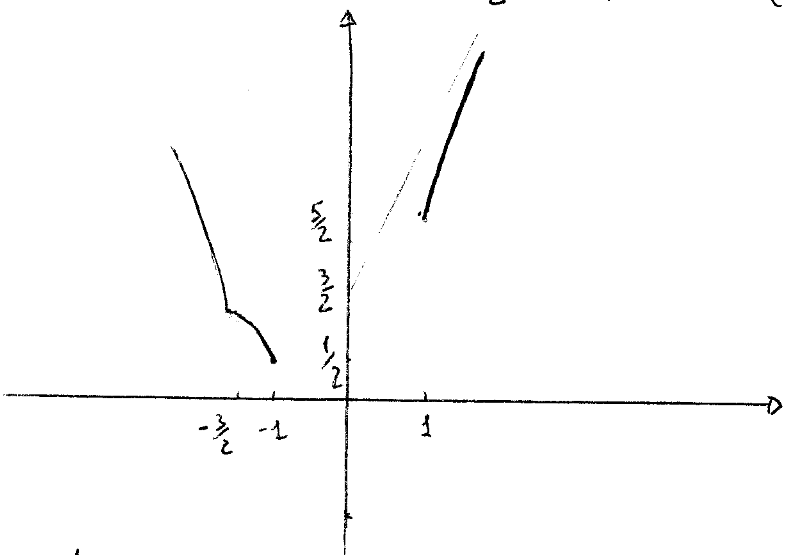


$(-1, \frac{1}{2})$
$(+1, \frac{5}{2})$
p.ti di minimo

$$y''>0 \iff \frac{\sqrt{x^2-1}-x\cdot\frac{2x}{2\sqrt{x^2-1}}}{x^2-1} = \frac{\cancel{x^2}-1-\cancel{x^2}}{(x^2-1)\sqrt{x^2-1}}>0 \iff \begin{cases} x^2-1<0 \\ x\in D \end{cases} \iff x\in\emptyset$$



funzione convessa in $(-\infty,-\frac{3}{2})$, $(-\frac{3}{2},-1)$ e in $(-1,+\infty)$.



4. $$\lim_{x\to+\infty}\frac{e^{\frac{1}{\sqrt{x}}}-e^{-\frac{1}{\sqrt{x}}}-2\,\text{arcsen}\frac{1}{\sqrt{x}}}{\text{sen}\frac{1}{x}-\frac{1}{x}} \underset{\frac{1}{\sqrt{x}}=y}{=} \lim_{y\to0^+}\frac{e^y-e^{-y}-2\,\text{arcsen}\,y}{\text{sen}\,y^2-y^2}=$$

$$=\lim_{y\to0^+}\frac{\cancel{1}+y+\cancel{\frac{y^2}{2}}+\frac{y^3}{3!}+\cancel{\frac{y^4}{4!}}+\frac{y^5}{5!}-(\cancel{1}-y+\cancel{\frac{y^2}{2}}-\frac{y^3}{3!}+\cancel{\frac{y^4}{4!}}-\frac{y^5}{5!})-2(y+\frac{y^3}{6}+\frac{3}{40}y^5)+o(y^5)}{\cancel{y^2}-\frac{y^6}{3!}+o(y^6)-\cancel{y^2}}$$

$$=\lim_{y\to0^+}\frac{\cancel{2y}+\cancel{\frac{2y^3}{6}}+\frac{2y^5}{120}-\cancel{2y}-\cancel{\frac{2y^3}{6}}-\frac{3}{20}y^5}{-\frac{y^6}{6}}=\lim_{y\to0^+}\frac{(\frac{1}{60}-\frac{3}{20})y^5}{-\frac{y^6}{6}}=+\infty$$

5. i) $\int \frac{1+tg^2x}{tgx+2}\sqrt{\frac{1-tgx}{1+tgx}}\,dx$

$t=\sqrt{\frac{1-tgx}{1+tgx}}$ $\quad 1-tgx = t^2(1+tgx)$

$tgx = \frac{1-t^2}{1+t^2}$ $\quad x = arctg\frac{1-t^2}{1+t^2}$

$$dx = \frac{1}{1+\left(\frac{1-t^2}{1+t^2}\right)^2}\,\frac{-2t(1+t^2)-2t(1-t^2)}{(1+t^2)^2}dt$$

$$dx = \frac{\cancel{(1+t^2)^2}}{\cancel{2}(1+t^4)}\cdot\frac{-\cancel{4}^{2}t}{\cancel{(1+t^2)^2}}dt$$

$$dx = -\frac{2t}{1+t^4}dt$$

$$=\int \frac{1+\left(\frac{1-t^2}{1+t^2}\right)^2}{\frac{1-t^2}{1+t^2}+2}\,t\cdot\frac{-2t}{1+t^4}dt$$

$$=-2\int \frac{2(\cancel{1+t^4})}{(1+t^2)^{\cancel{2}}}\,\frac{\cancel{1+t^2}}{1-t^2+2+2t^2}\cdot\frac{t^2}{\cancel{1+t^4}}dt$$

$$=-4\int\frac{t^2}{(t^2+1)(t^2+3)}dt$$

$$\frac{t^2}{(t^2+1)(t^2+3)}=\frac{At+B}{t^2+1}+\frac{Ct+D}{t^2+3}=\frac{At^3+Bt^2+3At+3B+Ct^3+Ct+Dt^2+D}{(t^2+1)(t^2+3)}$$

$$\Rightarrow\begin{cases}A+C=0\\B+D=1\\3A+C=0\\3B+D=0\end{cases}\Rightarrow\begin{cases}A=C=0\\B=-\frac12\\D=\frac32\end{cases}$$

$$I=-4\int\frac{-\frac12}{t^2+1}dt-4\int\frac{\frac32}{t^2+3}dt$$

$$=2\,arctg\,t-6\int\frac{1}{3\left(\left(\frac{t}{\sqrt3}\right)^2+1\right)}dt$$

$$=2\,arctg\,t-\frac{6}{\sqrt3}\,arctg\frac{t}{\sqrt3}+c$$

$$=2x-\frac{6}{\sqrt3}\,arctg\frac{tgx}{\sqrt3}+c.$$

ii) La funzione è definita e continua $\forall x$ t.c. $\begin{cases}\frac{1-tgx}{1+tgx}\ge0 \iff -1<tgx\le1\\ tgx\ne-2\end{cases}$

$\iff x\in\bigcup_{k\in\mathbb{Z}}\left]-\frac{\pi}{4}+k\pi,\frac{\pi}{4}+k\pi\right]$

La funzione è integ. in s.i. in $]-\frac{\pi}{4},0]$ perché $\lim_{x\to-\frac{\pi}{4}}h(x)=+\infty$ con ordine $\frac12<1$. Infatti $\lim_{x\to-\frac{\pi}{4}}\frac{1+tg^2x}{2+tgx}\cdot\sqrt{\frac{1-tgx}{1+tgx}}=\frac{2}{1}\cdot(+\infty)$ e

$$\lim_{x\to-\frac{\pi}{4}^+}\frac{1+tgx}{(x+\frac{\pi}{4})^\alpha}\overset{H}{=}\lim_{x\to-\frac{\pi}{4}^+}\frac{\frac{1}{\cos^2x}}{\alpha(x+\frac{\pi}{4})^{\alpha-1}}\in\mathbb{R}\text{ se }\alpha=1$$

Inoltre h è integrabile in s.i. in $[0,\frac{\pi}{4}[$, anzi lo è in senso classico perché continua in $[0,\frac{\pi}{4}]$ mentre non è integrabile in $[2,+\infty)$ non essendovi definita.