Corso di laurea in Chimica
Esame
ISTITUZIONI di MATEMATICHE I

18 luglio 2016

1. Studiare la seguente equazione nel campo complesso
$$z^6 - 4i = iz^6 + 4.$$

2. Studiare il dominio della funzione
$$y = \left(\frac{\sqrt{\ln 2x^2 - \ln x - \ln 2}}{\left(2^{x+1}\right)^2 - 2^x}\right)^x$$

3. Studiare la funzione
$$f(x) = |x+1|\, e^{\frac{1}{x}}$$
e disegnarne il grafico approssimativo.

4. Dire, giustificando le risposte, se la funzione $f(x)$ dell'esercizio precedente è integrabile in senso improprio in $[-1,0[$, $]0,1]$, $(-\infty,0[$ e $[1,+\infty)$.

5. Calcolare il seguente integrale indefinito
$$\int \frac{2\sqrt{x-1}+1}{\sqrt{x-1}} \arctan\sqrt{x-1}\,dx.$$

Svolgimento

1. $z^6 - i z^6 = 4i + 4$ $\qquad z^6 = 4\,\frac{1+i}{1-i}$ $\qquad z^6 = 4 \cdot \frac{(1+i)^2}{1-i^2}$ $\qquad z^6 = 4\,\frac{\cancel{1}+\cancel{i^2}+2i}{2}$

$$z = \sqrt[6]{4i} = \sqrt[6]{4\left(\cos\frac{\pi}{2} + i \operatorname{sen}\frac{\pi}{2}\right)} = \left\{ \sqrt[6]{4}\left(\cos\frac{\frac{\pi}{2}+2k\pi}{6} + i \operatorname{sen}\frac{\frac{\pi}{2}+2k\pi}{6}\right) \,\middle|\, k \in \{0,1,2,3,4,5\} \right\}$$

L'equazione ha 6 radici complesse a due a due opposte date da:

$$\omega_0 = \sqrt[3]{2}\left(\cos\frac{\pi}{12} + i \operatorname{sen}\frac{\pi}{12}\right) \qquad \omega_1 = \sqrt[3]{2}\left(\cos\frac{5}{12}\pi + i \operatorname{sen}\frac{5}{12}\pi\right)$$

$$\omega_2 = \sqrt[3]{2}\left(\cos\frac{\cancel{9}^{\,3}}{\cancel{12}_{\,4}}\pi + i \operatorname{sen}\frac{\cancel{9}^{\,3}}{\cancel{12}_{\,4}}\pi\right) \qquad \omega_3 = \sqrt[3]{2}\left(\cos\frac{13}{12}\pi + i \operatorname{sen}\frac{13}{12}\pi\right) = -\omega_0$$

$$\omega_4 = \sqrt[3]{2}\left(\cos\frac{17}{12}\pi + i \operatorname{sen}\frac{17}{12}\pi\right) = -\omega_1 \qquad \omega_5 = \sqrt[3]{2}\left(\cos\frac{21}{12}\pi + i \operatorname{sen}\frac{21}{12}\pi\right) = -\omega_2$$

2. $$\frac{\sqrt{\ln 2x^2 - \ln x - \ln 2}}{(2^{x+1})^2 - 2^x} > 0 \quad \vee \quad \begin{cases} \dfrac{\sqrt{\ln 2x^2 - \ln x - \ln 2}}{(2^{x+1})^2 - 2^x} = 0 \\ x > 0 \end{cases} \iff$$

$$\begin{cases} \ln 2x^2 - \ln x - \ln 2 > 0 \\ 2^{2x+2} > 2^x \end{cases} \vee \begin{cases} \ln 2x^2 - \ln x - \ln 2 = 0 \\ x > 0 \end{cases} \iff$$

$$\begin{cases} \cancel{\ln 2} + \ln x^2 - \ln x - \cancel{\ln 2} > 0 \\ 2x + 2 > x \end{cases} \vee \begin{cases} \cancel{\ln 2} + \ln x^2 - \ln x - \ln 2 = 0 \\ x > 0 \end{cases} \iff$$

$$\begin{cases} \ln x^2 > \ln x \\ x > -2 \end{cases} \vee \begin{cases} \ln x^2 = \ln x \\ x > 0 \end{cases} \iff \begin{cases} x > 0 \\ x^2 > x \\ x > -2 \end{cases} \vee \begin{cases} x > 0 \\ x^2 = x \\ x > 0 \end{cases} \iff$$

$$\begin{cases} x > 0 \\ x > 1 \\ x > -2 \end{cases} \vee \begin{cases} x > 0 \\ x = 1 \end{cases} \iff x > 1 \vee x = 1 \qquad ID = [1, +\infty)$$

3. $ID = R - \{0\}$ $\qquad f(-x) \neq \pm f(x)$ $\qquad f$ né pari né dispari

$f(x) = 0 \iff x + 1 = 0 \iff x = -1$ $\qquad (-1, 0)$ intersezione con asse x

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$f(x)>0 \iff |x+1|e^{\frac{1}{x}}>0 \iff |x+1|>0 \iff \begin{array}{l} x\neq -1 \\ x\in\mathbb{R}^* \end{array}$

$\lim_{x\to 0^-} |x+1|e^{\frac{1}{x}} = 0$ poiché $\lim_{x\to 0^-} e^{\frac{1}{x}} \overset{\frac{1}{x}=y}{=} \lim_{y\to -\infty} e^{y} = 0$

$\lim_{x\to 0^+} |x+1|e^{\frac{1}{x}} = +\infty$ poiché $\lim_{x\to 0^+} e^{\frac{1}{x}} = \lim_{y\to +\infty} e^{y} = +\infty$

$x=0$ As. verticale solo da destra,

$\lim_{x\to \pm\infty} |x+1|e^{\frac{1}{x}} = +\infty$ poiché $\lim_{x\to \pm\infty} e^{\frac{1}{x}} \overset{\frac{1}{x}=y}{=} \lim_{y\to 0} e^{y} = 1$

$\nexists$ As. orizz. a $\pm\infty$

$\lim_{x\to +\infty} \frac{|x+1|e^{\frac{1}{x}}}{x} = \lim_{x\to +\infty} \frac{x+1}{x} e^{\frac{1}{x}} = 1$

$\lim_{x\to +\infty} |x+1|e^{\frac{1}{x}} - x = \lim_{x\to +\infty} (x+1)e^{\frac{1}{x}} - x = \lim_{x\to +\infty} x(e^{\frac{1}{x}}-1) + e^{\frac{1}{x}} =$

$\left(\text{poiché } \begin{array}{c} e^{y}-1\sim y \\ \text{se } y\to 0 \end{array}\right) = \lim_{x\to +\infty} \cancel{x}\cdot\frac{1}{\cancel{x}} + e^{\frac{1}{x}} = 2$ $\quad y = x+2$ As. obliquo a $+\infty$

$\lim_{x\to -\infty} \frac{|x+1|e^{\frac{1}{x}}}{x} = \lim_{x\to -\infty} \frac{-x-1}{x} e^{+\frac{1}{x}} = -1$

$\lim_{x\to -\infty} |x+1|e^{\frac{1}{x}} + x = \lim_{x\to -\infty} (-x-1)e^{\frac{1}{x}} + x = \lim_{x\to -\infty} -x(e^{\frac{1}{x}}-1) - e^{\frac{1}{x}}$

$\left(\text{poiché } \begin{array}{c} e^{y}-1\sim y \\ \text{se } y\to 0 \end{array}\right) = \lim_{x\to -\infty} -\cancel{x}\cdot\frac{1}{\cancel{x}} - e^{\frac{1}{x}} = -2$ $\quad y=-x-2$ As. obl. a $-\infty$

$y' = \frac{x+1}{|x+1|} e^{\frac{1}{x}} + |x+1|\, e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) = \left[(x+1)e^{\frac{1}{x}} - \frac{|x+1|^2 e^{\frac{1}{x}}}{x^2}\right]\frac{1}{|x+1|}$

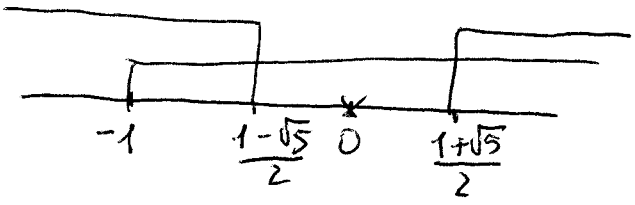
$= \frac{x+1}{|x+1|} e^{\frac{1}{x}}\left(1 - \frac{x+1}{x^2}\right) \quad \forall x\in\mathbb{R}^* \quad x\neq -1$

Poiché $\lim_{x\to -1^\mp} y' = \lim_{x\to -1^\mp} \mp e^{\frac{1}{x}}\left(1-\frac{x+1}{x^2}\right) = \mp\frac{1}{e}$ si ha $x=-1$ pto angoloso essendo $f'_-(-1) = -\frac{1}{e}$ $f'_+(-1) = \frac{1}{e}$.

$$f'(x)>0 \iff (x+1)\left(1-\frac{x+1}{x^2}\right)>0 \iff \begin{cases} x>-1 \wedge x\neq 0 \\ \dfrac{x^2-x-1}{x^2}>0 \end{cases} \vee \begin{cases} x<-1 \\ \dfrac{x^2-x-1}{x^2}<0 \end{cases}$$

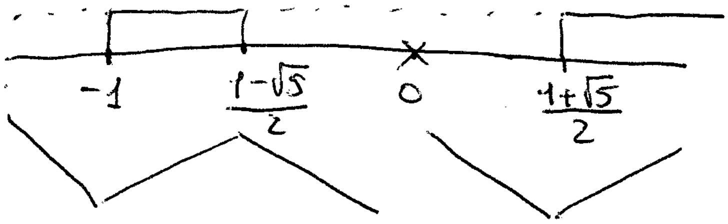
$$\iff \begin{cases} x>-1 \wedge x\neq 0 \\ x^2-x-1>0 \end{cases} \vee \begin{cases} x<-1 \\ x^2-x-1<0 \end{cases} \iff \begin{cases} x>-1 \wedge x\neq 0 \\ x<\frac{1-\sqrt5}{2} \vee x>\frac{1+\sqrt5}{2} \end{cases} \vee \begin{cases} x<-1 \\ \frac{1-\sqrt5}{2}<x<\frac{1+\sqrt5}{2} \end{cases}$$

$\frac{1\pm\sqrt5}{2}$



$-1<x<\frac{1-\sqrt5}{2} \vee x>\frac{1+\sqrt5}{2} \vee x\in\emptyset$

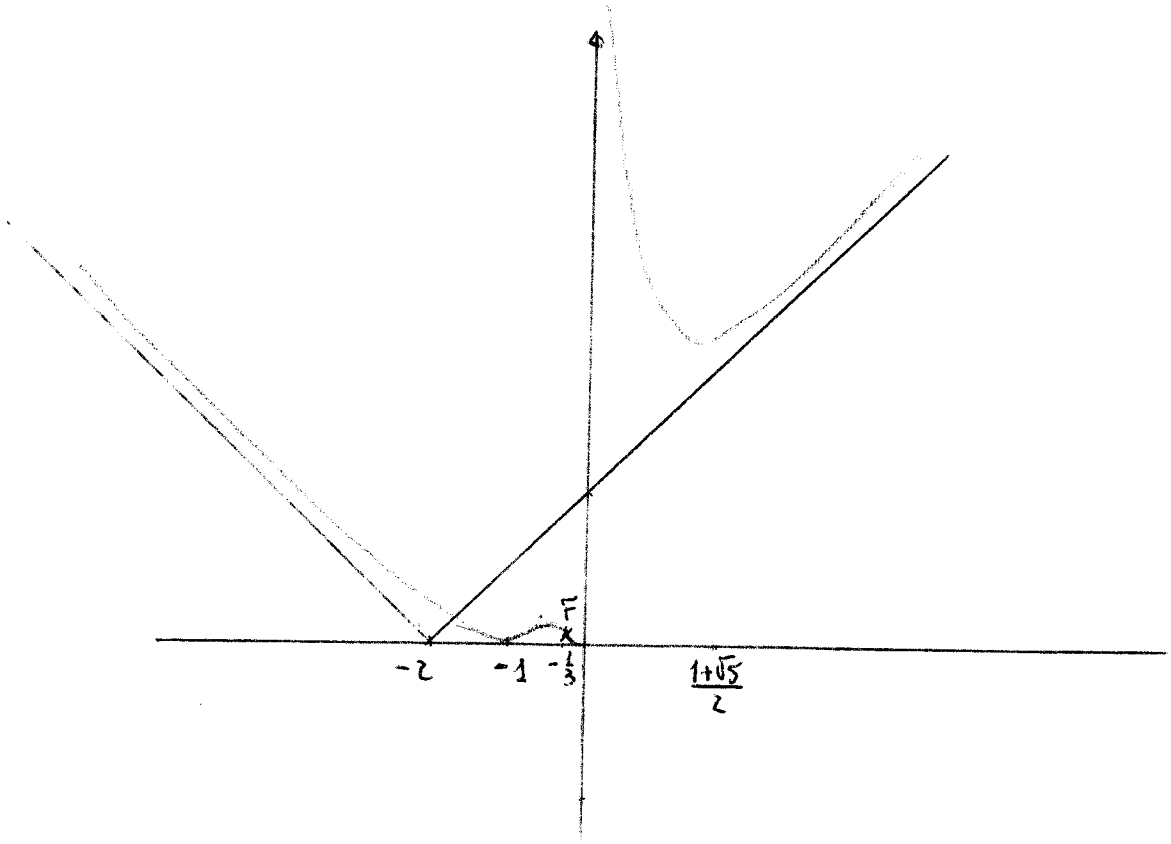
$x\neq 0$



$\begin{cases} x=-1 \\ y=0 \end{cases}$ minimo rel. (e assoluto)

$x=\frac{1-\sqrt5}{2}$ max rel

$x=\frac{1+\sqrt5}{2}$ min rel

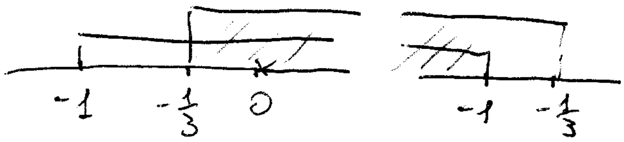


Poiché $y'=\begin{cases} e^{\frac{1}{x}}\left(1-\dfrac{x+1}{x^2}\right) & \text{se } x>-1 \wedge x\neq 0 \\ -e^{\frac{1}{x}}\left(1-\dfrac{x+1}{x^2}\right) & \text{se } x<-1 \end{cases}$

segue che

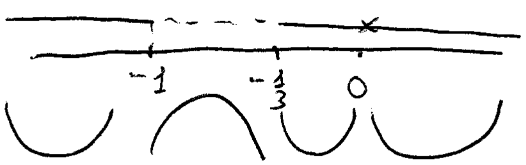
$$y'' = \begin{cases} e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)\left(1-\frac{x+1}{x^2}\right)+e^{\frac{1}{x}}\left(-\frac{x^2-2x(x+1)}{x^4}\right)=e^{\frac{1}{x}}\,\frac{\cancel{-x^2}+x+1+\cancel{x^2}+2x}{x^4} & x>-1 \wedge x\neq 0 \\ -\left[e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)\left(1-\frac{x+1}{x^2}\right)+e^{\frac{1}{x}}\left(-\frac{x^2-2x(x+1)}{x^4}\right)\right]=-e^{\frac{1}{x}}\,\frac{3x+1}{x^4} & x<-1 \end{cases}$$

$$y''>0 \iff \begin{cases} x>-1 \wedge x\neq 0 \\ 3x+1>0 \end{cases} \vee \begin{cases} x<-1 \\ 3x+1 \not> 0 \end{cases}$$



$x>-\frac{1}{3} \wedge x\neq 0 \vee x<-1$

$y''>0 \iff x<-1 \vee x>-\frac{1}{3},\ x\neq 0$



f è convessa per $x<-1$; $-\frac{1}{3}<x<0$, $x>0$.

f è concava per $-1<x<-\frac{1}{3}$

$x=-\frac{1}{3}$ pto di flesso

4. Poiché f è continua in $\mathbb{R}^*$ e

$\lim_{x\to 0^-} |x+1|\,e^{\frac{1}{x}} = 0 \Rightarrow$ f è integrabile in senso classico (e quindi anche in senso improprio) in $[-1,0[$.

$\lim_{x\to 0^+} |x+1|\,e^{\frac{1}{x}} = +\infty$ con ordine comunque grande $\Rightarrow$ f non è integrabile in s. improprio in $]0,1]$.

$\lim_{x\to -\infty} |x+1|\,e^{\frac{1}{x}} = +\infty$ $(\neq 0)$ $\Rightarrow$ f non è integ. in senso improprio in $]-\infty,0[$

$\lim_{x\to +\infty} |x+1|\,e^{\frac{1}{x}} = +\infty$ $(\neq 0)$ $\Rightarrow$ f non è integ. in senso improprio in $[1,+\infty)$.

5. Posto $\sqrt{x-1}=t \Rightarrow x=t^2+1 \Rightarrow dx = 2t\,dt$ si ha

$$\int \frac{2\sqrt{x-1}+1}{\sqrt{x-1}}\,\mathrm{arctg}\sqrt{x-1}\,dx = \int \frac{2t+1}{\cancel{t}}\,\mathrm{arctg}\,t \cdot 2\cancel{t}\,dt = 2\int D(t^2+t)\,\mathrm{arctg}\,t\,dt$$

$$= 2(t^2+t)\,\mathrm{arctg}\,t - 2\int \frac{t^2+t}{t^2+1}\,dt = 2(t^2+t)\,\mathrm{arctg}\,t - 2\int\left(1+\frac{t-1}{t^2+1}\right)dt$$

$$= 2(t^2+t)\,\mathrm{arctg}\,t - 2\left(t+\frac{1}{2}\lg(t^2+1)-\mathrm{arctg}\,t\right)+c = 2\left(x-1+\sqrt{x-1}\right)\mathrm{arctg}\sqrt{x-1} - 2\sqrt{x-1} - \lg x + 2\,\mathrm{arctg}\sqrt{x-1} + c.$$