Corso di laurea triennale in Chimica
Esame-Esonero di
Istituzioni di Matematiche I Traccia A

30 Gennaio 2015

1. Calcolare e disegnare poi nel piano complesso le radici dell'equazione

$$|z|^2\,\bar{z}^4 = \left(\frac{1-i}{1+i}\right)^{13}\left(\cos\frac{3}{8}\pi + i\sin\frac{3}{8}\pi\right)^8.$$

2. Studiare la disequazione

$$\frac{\lg_{0,5}\left(3+x-\sqrt{2x^2-x-1}\right)+1}{(0,5)^{x-2}-4(0,5)^{|x|}-1} > 0$$

3. Data la funzione
$$g(x) = \left(\sqrt[4]{x^4+x} - \sqrt[3]{x^3+x}\right)\sin\frac{1}{x},$$
dire, giustificando la risposta, se è integrabile in senso classico o in senso improprio in $]0,1]$, in $[1,2]$ e in $[2,+\infty)$.

4. Calcolare il seguente integrale indefinito

$$\int\left(x^2+x\right)\arctan(2x+1)dx.$$

5. Studiare la funzione

$$f(x) = |x-1| - \ln\left(\frac{x+3}{x+1}\right)$$

e disegnarne il grafico qualitativo.

N.B. Chi ha superato il primo esonero deve fare solo gli ultimi 3 esercizi.

Svolgimento

1. Poiché $\left(\frac{1-i}{1+i}\right)^{13} = \left[\frac{(1-i)^2}{1-i^2}\right]^{13} = \left(\frac{\cancel{1}+\cancel{i^2}-2i}{2}\right)^{13} = (-i)^{13} = \left((-i)^4\right)^3(-i) = -i$

mentre $\left(\cos \frac{3}{8}\pi + i \operatorname{sen} \frac{3}{8}\right)^8 = \cos 3\pi + i \operatorname{sen} 3\pi = -1$

l'equazione diventa $|z|^2 \bar{z}^4 = (-i)(-1)$

Posto $z = \rho(\cos\vartheta + i \operatorname{sen}\vartheta) \Rightarrow |z| = \rho \wedge \bar{z} = \rho(\cos(-\vartheta) + i \operatorname{sen}(-\vartheta))$

si ha $\rho^2(\rho^4(\cos(-4\vartheta) + i \operatorname{sen}(-4\vartheta)) = i$ cioè

$\rho^6(\cos(-4\vartheta) + i \operatorname{sen}(-4\vartheta)) = \cos\frac{\pi}{2} + i \operatorname{sen}\frac{\pi}{2} \iff$

$\rho^6 = 1 \wedge \exists k \in \mathbb{Z}$ t.c. $-4\vartheta = \frac{\pi}{2} + 2k\pi \iff$

$$\begin{cases} \rho = 1 \\ \vartheta = -\frac{\pi}{8} + k\frac{\pi}{2} \quad k \in \mathbb{Z} \end{cases}$$

Per $k = 0$ $\quad \omega_0 = \cos\left(-\frac{\pi}{8}\right) + i \operatorname{sen}\left(-\frac{\pi}{8}\right)$

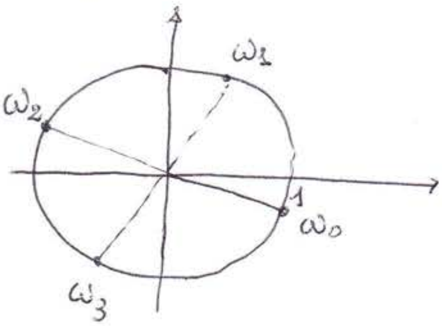
$k = 1$ $\quad \omega_1 = \cos\left(-\frac{\pi}{8} + \frac{\pi}{2}\right) + i \operatorname{sen}\left(-\frac{\pi}{8} + \frac{\pi}{2}\right) = \cos\frac{3}{8}\pi + i \operatorname{sen}\frac{3}{8}\pi$

$k = 2$ $\quad \omega_2 = \cos\left(-\frac{\pi}{8} + \pi\right) + i \operatorname{sen}\left(-\frac{\pi}{8} + \pi\right) = -\omega_0$

$k = 3$ $\quad \omega_3 = \cos\left(-\frac{\pi}{8} + \frac{3}{2}\pi\right) + i \operatorname{sen}\left(-\frac{\pi}{8} + \frac{3}{2}\pi\right) = \cos\left(\frac{11}{8}\pi\right) + i \operatorname{sen}\left(\frac{11}{8}\pi\right)$

$= \cos\left(\frac{3}{8}\pi + \pi\right) + i \operatorname{sen}\left(\frac{3}{8}\pi + \pi\right) = -\omega_1$

Per $k \in \mathbb{Z} - \{0, 1, 2, 3\}$ si riottengono sempre le stesse soluzioni.



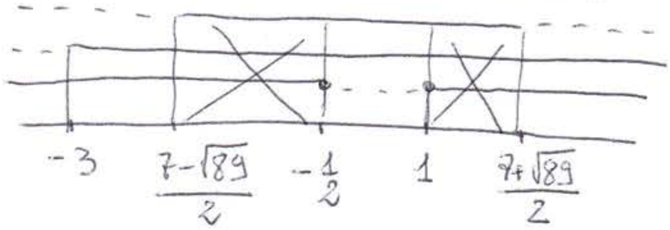
2. Prima di tutto studiamo il dominio della frazione:

$$3+x-\sqrt{2x^2-x-1}>0 \iff \sqrt{2x^2-x-1}<3+x \iff \begin{cases}2x^2-x-1\ge 0\\ x+3>0\\ 2x^2-x-1<x^2+9+6x\end{cases}$$

$$x^2-7x-10<0$$

$$x_{1/2}=\frac{1\pm 3}{2}=\left\langle\begin{matrix}-\frac{1}{2}\\ 1\end{matrix}\right. \qquad \begin{cases}x\le -\frac{1}{2} \ \vee \ x\ge 1\\ x>-3\\ \frac{7-\sqrt{89}}{2}<x<\frac{7+\sqrt{89}}{2}\end{cases}$$

$$x_{1/2}=\frac{7\pm\sqrt{89}}{2}$$

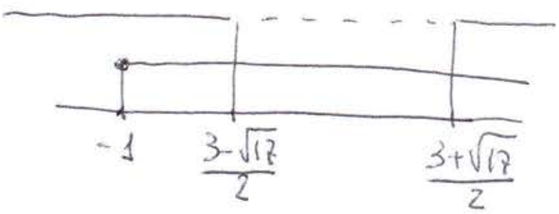


$N>0 \qquad \log_{0,5}\left(3+x-\sqrt{2x^2-x-1}\right)>-1 \iff 0<3+x-\sqrt{2x^2-x-1}<2$

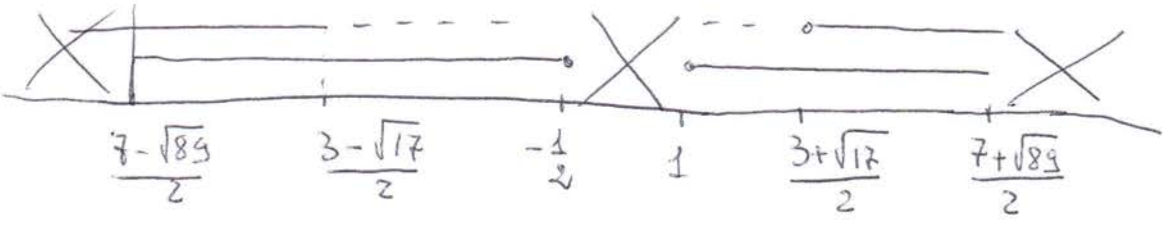
$$\iff \begin{cases}\sqrt{2x^2-x-1}<3+x\\ x+1<\sqrt{2x^2-x-1}\end{cases} \iff \begin{cases}\frac{7-\sqrt{89}}{2}<x\le-\frac{1}{2} \ \vee \ 1\le x<\frac{7+\sqrt{89}}{2}\\ \begin{cases}x+1<0\\ 2x^2-x-1\ge 0\end{cases} \vee \begin{cases}x+1\ge 0\\ (x+1)^2<2x^2-x-1\end{cases}\end{cases}$$

$$\begin{cases}x<-1\\ x\le-\frac{1}{2} \ \vee \ x\ge 1\end{cases} \vee \begin{cases}x\ge -1\\ x^2-3x-2>0\end{cases}$$

$$x<-1 \qquad x<\frac{3-\sqrt{17}}{2} \ \vee \ x>\frac{3+\sqrt{17}}{2}$$



$$N>0 \iff \begin{cases}x<\frac{3-\sqrt{17}}{2} \ \vee \ x>\frac{3+\sqrt{17}}{2}\\ \frac{7-\sqrt{89}}{2}<x\le-\frac{1}{2} \ \vee \ 1\le x<\frac{7+\sqrt{89}}{2}\end{cases}$$

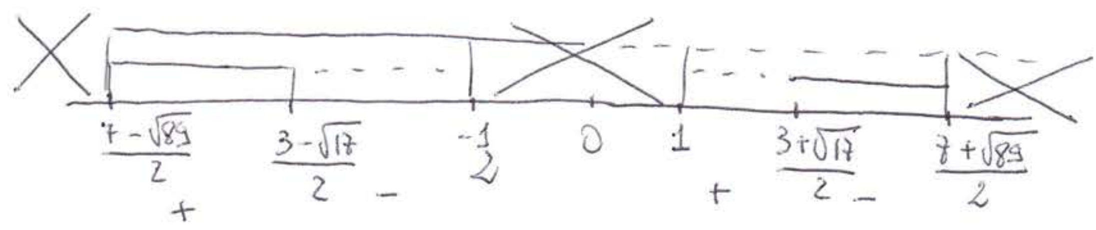


$$D>0 \iff \begin{cases}x\ge 0\\ 4\left(\cancel{\tfrac{1}{2}}\right)^x-4\left(\cancel{\tfrac{1}{2}}\right)^x-1>0\end{cases} \vee \begin{cases}x<0\\ 4\left(\frac{1}{2}\right)^x-4\cdot 2^x-1>0\end{cases} \qquad 4-4\cdot 2^{2x}-2^x>0$$

$\varnothing$

Posto $2^x=t \qquad 4t^2-t-4<0 \qquad t_{1/2}=\frac{1\pm\sqrt{65}}{8}$

$$\underset{\overset{\wedge}{0}}{\frac{1-\sqrt{65}}{2}}<2^x<\frac{1+\sqrt{65}}{2} \qquad \begin{cases}x<0\\ x<\log_2\left(\frac{1+\sqrt{65}}{2}\right)\end{cases} \iff x<0$$



la disequazione è soddisfatta $\forall x \in \left]\frac{7-\sqrt{89}}{2}, \frac{3-\sqrt{17}}{2}\right[\cup \left[1, \frac{3+\sqrt{17}}{2}\right[$.

4. $\int (x^2+x)\operatorname{arctg}(2x+1)\,dx = \int D\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\operatorname{arctg}(2x+1)\,dx =$ integro per parti

$$\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\operatorname{arctg}(2x+1) - \int\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\frac{1}{1+(2x+1)^2}\cdot 2\,dx.$$

Poiché $\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\frac{2}{1+(2x-1)^2} = \frac{1}{6}\,\frac{2x^3+3x^2}{4x^2+4x+2}\cdot 2 = \frac{1}{6}\,\frac{2x^3+3x^2}{2x^2+2x+1}$

$= \frac{1}{6}\left(x+\frac{1}{2}+\frac{2x-\frac{1}{2}}{2x^2-2x+1}\right)$ essendo

$$\begin{array}{l|l} 2x^3+3x^2 & 2x^2+2x+1 \\ \hline 2x^3+2x^2+x & x+\frac{1}{2} \\ // \quad +x^2-x & \\ \quad +x^2+x+\frac{1}{2} & \\ // \quad -2x-\frac{1}{2} & \end{array}$$

il secondo integrale diventa

$$\frac{1}{6}\int\left(x+\frac{1}{2}-\frac{2x-\frac{1}{2}}{2x^2+2x+1}\right)dx$$

$$= \frac{1}{6}\left[\frac{x^2}{2}+\frac{1}{2}x-\frac{1}{2}\int\frac{4x+1}{2x^2+2x+1}\,dx\right] = \frac{1}{6}\left[\frac{x^2}{2}+\frac{1}{2}x-\frac{1}{2}\int\left(\frac{4x+2}{2x^2+2x+1}+\frac{1}{2x^2-2x+1}\right)dx\right]$$

$$= \frac{1}{6}\left[\frac{x^2}{2}+\frac{1}{2}x-\frac{1}{2}\lg(2x^2+2x+1)+\frac{1}{2}\int\frac{2}{4x^2-2x+2}\,dx\right]$$

$$= \frac{x^2}{12}+\frac{x}{12}-\frac{1}{12}\lg(2x^2+2x+1)+\frac{1}{12}\operatorname{arctg}(2x-1)+c$$

Concludendo

$$\int(x^2+x)\operatorname{arctg}(2x+1)\,dx = \left(\frac{x^3}{3}+\frac{x^2}{2}\right)\operatorname{arctg}(2x+1)-\frac{x^2}{12}+\frac{x}{12}+$$

$$+\frac{1}{12}\lg(2x^2+2x+1)+\frac{1}{12}\operatorname{arctg}(2x+1)+c$$

3. ID: $\begin{cases} x^4+x \geq 0 \\ x \neq 0 \end{cases}$ $\begin{cases} x(x^3+1) \geq 0 \\ x \neq 0 \end{cases}$ $x \leq -1 \vee x > 0$

Poiché f è definita e continua in $]0,1]$ e

$\lim_{x\to 0} \left(\sqrt[4]{x^4+x} - \sqrt[3]{x^3+x}\right) \operatorname{sen}\frac{1}{x} = 0$ perché prodotto di una funzione infinitesima per una limitata,

g è integrabile in senso improprio (anzi in senso classico se la prolunghiamo in $x=0$) in $]0,1]$. Inoltre g è integrabile in senso classico in $[1,2]$ perché continua in un intervallo chiuso e limitato. Infine in $[2,+\infty)$ g è integrabile in senso improprio perché continua e infinitesima a $+\infty$ con ordine $2>1$ essendo

$$\lim_{x\to+\infty} \left(\sqrt[4]{x^4+x} - \sqrt[3]{x^3+x}\right) \operatorname{sen}\frac{1}{x} = \lim_{x\to+\infty} x\left(\sqrt[4]{1+\frac{1}{x^3}} - \sqrt[3]{1+\frac{1}{x^2}}\right) \operatorname{sen}\frac{1}{x}$$

$$= \lim_{x\to+\infty} \left(\left(1+\frac{1}{x^3}\right)^{\frac{1}{4}} - \left(1+\frac{1}{x^2}\right)^{\frac{1}{3}}\right) \underbrace{\frac{\operatorname{sen}\frac{1}{x}}{\frac{1}{x}}}_{1} = \lim \left[\cancel{1} + \overbrace{\frac{1}{4x^3}}^{0 \text{ con ord } 3} - \cancel{1} - \overbrace{\frac{1}{x^2}}^{0 \text{ con ord } 2}\right] =$$

$= \lim_{x\to+\infty} -\frac{1}{x^2} = 0$ con ordine 2. $(1+x)^a \sim 1+\alpha x$ per $x \to 0$

4. ID: $\frac{x+3}{x+1} > 0 \iff x < -3 \vee x > -1$ $ID = (-\infty,-3[\cup]-1,+\infty)$

f non è né pari né dispari perché il suo dominio non è simmetrico rispetto all'origine.

$\begin{cases} x = 0 \\ y = 1 - \ln 3 < 0 \end{cases}$ $\begin{cases} y = 0 \\ |x-1| = \ln\left(\frac{x+3}{x+1}\right) \end{cases}$ non studio intersezioni con asse x né positività (servirebbe metodo grafico)

$\lim_{x\to-3^-} f(x) = 4 - \lim_{x\to-3^-} \ln\frac{x+3}{x+1} = 4 - \lim_{y\to0^+} \ln y = +\infty$

$\lim_{x\to-1^+} f(x) = 2 - \lim_{x\to-1^+} \ln\frac{x+3}{x+1} = 4 - \lim_{y\to+\infty} \ln y = -\infty$

$x=-3$ As. $x=-1$ vert.

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} |x-1| - \ln \underbrace{\frac{x+3}{x+1}}_{\ln 1 = 0} = +\infty$$ $\nexists$ as. orizz. a $\pm\infty$

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x-1}{x} - \frac{\ln \frac{x+3}{x+1}}{x} = 1$$

$$\lim_{x\to+\infty} f(x) - x = \lim_{x\to+\infty} \cancel{x} - 1 - \ln \underbrace{\frac{x+3}{x+1}}_{0} - \cancel{x} = -1$$

$y = x - 1$ As. obliquo a $+\infty$

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{-x+1}{x} - \frac{\ln \frac{x+3}{x+1}}{x} = -1$$

$y = -x + 1$ As. obliquo a $-\infty$

$$\lim_{x\to-\infty} f(x) + x = \lim_{x\to-\infty} -\cancel{x} + 1 - \ln \underbrace{\frac{x+3}{x+1}}_{\to 0} + \cancel{x} = 1$$

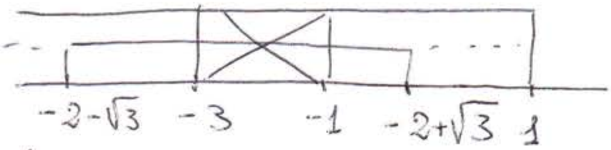
$$y' = \frac{x-1}{|x-1|} - \frac{\cancel{x+1}}{x+3} \cdot \frac{\cancel{x}+1-\cancel{x}-3}{(x+1)^{\cancel{2}}} = \frac{x-1}{|x-1|} + \frac{2}{(x+3)(x+1)} = \begin{cases} 1 + \frac{2}{(x+3)(x+1)} & \text{se } x > 1 \\ -1 + \frac{2}{(x+3)(x+3)} & \text{se } x < 1 \end{cases}$$

$$y' > 0 \iff \begin{cases} x > 1 \\ 1 + \frac{2}{(x+3)(x+1)} > 0 \end{cases} \quad \lor \quad \begin{cases} x < 1 \\ 1 - \frac{2}{(x+3)(x+1)} > 0 \end{cases}$$

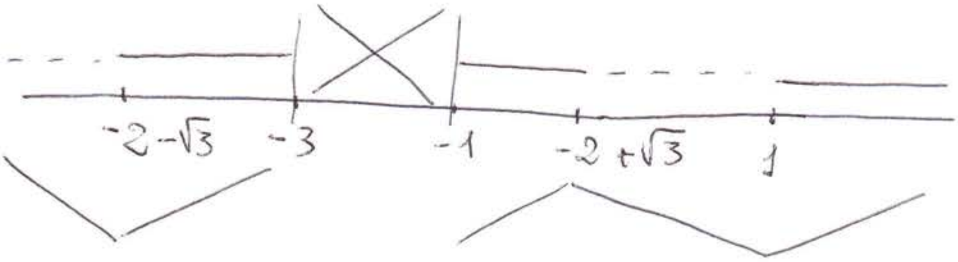
$\forall x > 1$

$x^2 + 4x + 1 < 0 \qquad x_{1/2} = -2 \pm \sqrt{3}$

$-2-\sqrt{3} < x < -2+\sqrt{3}$



$y' > 0$ per $-2-\sqrt{3} < x < -3 \;\lor\; -1 < x < -2+\sqrt{3} \;\lor\; x > 1$



$$m \begin{cases} x = -2-\sqrt{3} \\ y = f(-2-\sqrt{3}) = 3 + \sqrt{3} - \ln\left(\frac{1-\sqrt{3}}{-1-\sqrt{3}}\right) = 3 + \sqrt{3} - \ln \frac{\sqrt{3}-1}{\sqrt{3}+1} > 0 \end{cases}$$

$$M \begin{cases} x = -2+\sqrt{3} \\ y = f(-2+\sqrt{3}) = 3-\sqrt{3} - \ln\left(\dfrac{1+\sqrt{3}}{\sqrt{3}-1}\right) = 3-\sqrt{3} - \ln(2+\sqrt{3}) < 0 \end{cases}$$

$$m \begin{cases} x = 1 \\ y = -\ln 2 < 0 \end{cases}$$

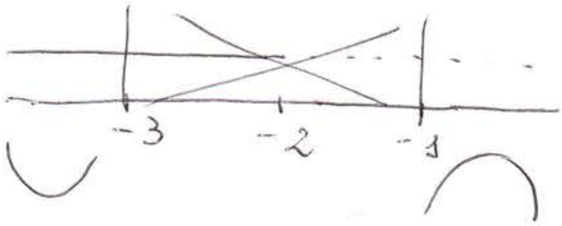
$x = 1$ pto angoloso perché

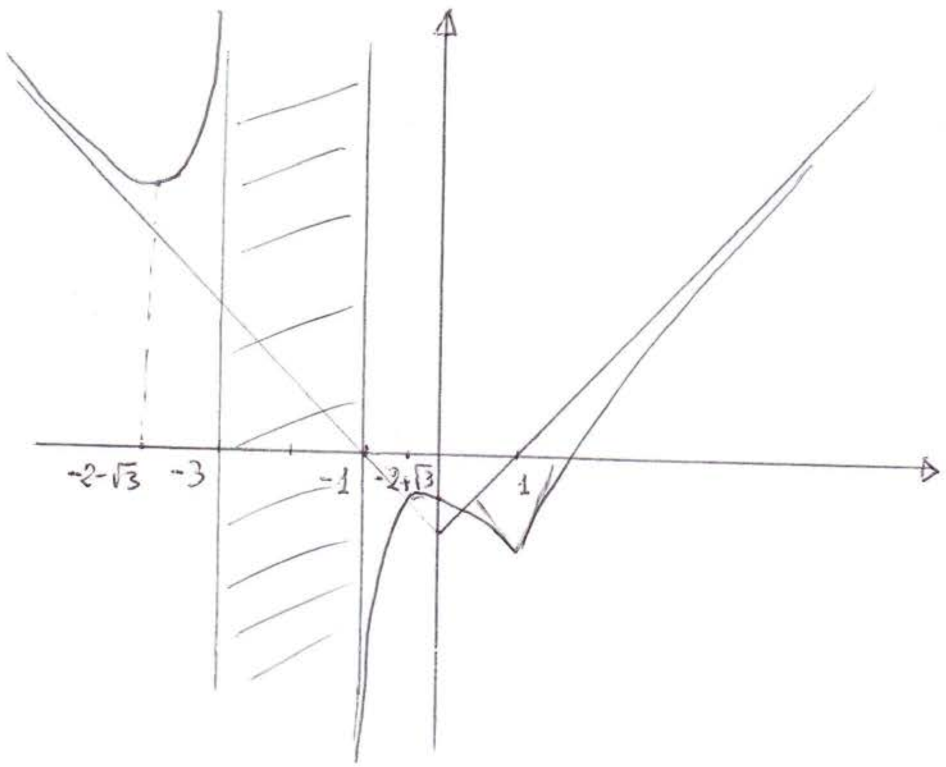
$$\lim_{x\to 1^-} y' = \lim_{x \to 1^-} -1 + \frac{2}{(x+3)(x+1)} = -\frac{3}{4}$$

$$\lim_{x\to 1^+} y' = \lim_{x \to 1^+} 1 + \frac{2}{(x+3)(x+1)} = \frac{5}{4}$$

$$y'' = D\left(\pm 1 + \frac{2}{(x+3)(x+1)}\right) = -2\,\frac{2x+4}{(x+3)^2(x+1)^2} \qquad \forall x \in ID,\ x \neq 1$$

$y'' > 0 \iff x + 2 < 0 \qquad x < -2$





Dal grafico si deduce a posteriori che f è positiva per $x < -3$.