COMPITO DI ISTITUZIONI DI MATEMATICHE I
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Esercizio 1. Studiare la funzione

$$\operatorname{arctang}\left(\frac{2x+1-|x|}{|x+1|}\right)$$

e disegnarne il grafico approssimativo.

Esercizio 2. Stabilire se la seguente funzione

$$\frac{\log(x^2+1)}{e^x-1}$$

è integrabile negli intervalli $]0,1]$, $[1,\infty[$.

Esercizio 3. Senza utilizzare il Teorema di de l'Hopital calcolare il seguente limite

$$\lim_{x\to 0}\frac{\sqrt{e^x-1}\,\log(\sqrt{x}+1)-\operatorname{arctang}(\operatorname{sen}(x))}{\operatorname{sen}(\log(\operatorname{sen}(x)+1))}.$$

Esercizio 4. Calcolare il seguente integrale indefinito

$$\int\frac{dx}{\cos(x)+2\operatorname{sen}(x)+1}.$$

Esercizio 5. Risolvere la seguente equazione in campo complesso

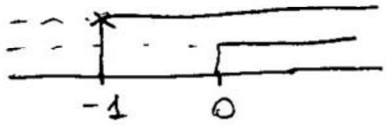
$$w^2-2iw+i=0.$$

Svolgimento

1. $f(x) = \operatorname{arctg}\left(\dfrac{2x+1-|x|}{|x+1|}\right)$ I.D.: $x+1 \neq 0$ $\quad \mathbb{R}-\{-1\}$

$x \geq 0$

$x+1>0 \quad x>-1$



$$f(x) = \begin{cases} \operatorname{arctg} \dfrac{2x+1+x}{-x-1} = -\operatorname{arctg} \dfrac{3x+1}{x+1} & \text{se } x<-1 \\ \operatorname{arctg} \dfrac{2x+1+x}{x+1} = \operatorname{arctg} \dfrac{3x+1}{x+1} & \text{se } -1<x<0 \\ \operatorname{arctg} \dfrac{2x+1-x}{x+1} = \operatorname{arctg} 1 = \dfrac{\pi}{4} & \text{se } x \geq 0 \end{cases}$$

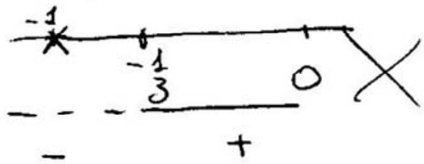
Dunque per $x \geq 0$ $f(x) = \frac{\pi}{4}$ e basta perciò studiare f per $x<0$.

$$f(x)>0 \iff \begin{cases} -\operatorname{arctg} \dfrac{3x+1}{x+1} > 0 \\ x<-1 \end{cases} \vee \begin{cases} \operatorname{arctg} \dfrac{3x+1}{x+1} > 0 \\ -1<x<0 \end{cases} \iff \begin{cases} \dfrac{3x+1}{x+1} < 0 \\ x<-1 \end{cases} \vee \begin{cases} \dfrac{3x+1}{x+1} > 0 \\ -1<x<0 \end{cases}$$

$$\iff \begin{cases} 3x+1>0 \\ x<-1 \end{cases} \vee \begin{cases} 3x+1>0 \\ -1<x<0 \end{cases} \iff \begin{cases} x>-\frac{1}{3} \\ x<-1 \end{cases} \vee \begin{cases} x>-\frac{1}{3} \\ -1<x<0 \end{cases} \iff -\frac{1}{3}<x<0$$

$\emptyset$

$f(x)>0$ per $-\frac{1}{3}<x<0$



$f(x)=0 \iff x=-\frac{1}{3}$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} -\operatorname{arctg} \dfrac{3x+1}{x+1} = -\operatorname{arctg} 3 \qquad y=-\operatorname{arctg} 3$ As. orizz.

$$\left.\begin{aligned} &\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} -\operatorname{arctg} \frac{3x+1}{x+1} = -\lim_{y\to+\infty} \operatorname{arctg} y = -\frac{\pi}{2} \\ &\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} \operatorname{arctg} \frac{3x+1}{x+1} = \lim_{y\to-\infty} \operatorname{arctg} y = -\frac{\pi}{2} \end{aligned}\right\} \Rightarrow \lim_{x\to-1} f(x) = \frac{\pi}{2}$$

$\nexists$ as. verticale

se $x<-1$

$$y' = -\frac{1}{1+\left(\frac{3x+1}{x+1}\right)^2} \cdot \frac{3(x+1)-3x-1}{(x+1)^2} = -\frac{2}{(x+1)^2+(3x+1)^2} = -\frac{2}{10x^2+8x+2}$$

se $-1<x<0 \qquad y' = \dfrac{2}{10x^2+8x^2+2}$

$$y'>0 \iff \begin{cases} x<-1 \\ -\dfrac{\cancel{2}}{\cancel{2}(5x^2+4x+1)} > 0 \end{cases} \vee \begin{cases} -1<x<0 \\ \dfrac{1}{5x^2+4x+1} > 0 \end{cases} \iff -1<x<0$$

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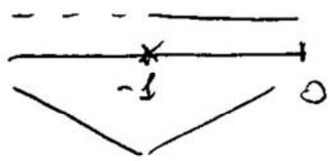


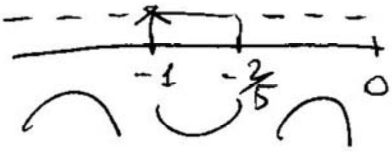
$\lim\limits_{x\to-1^\mp} y' = \mp\dfrac{2}{4} = \mp\dfrac{1}{2}$

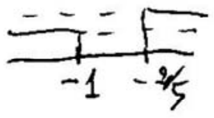
se $\quad x<-1$

$$y''=D\left(-\frac{1}{5x^2+4x+1}\right)=\frac{10x+4}{(5x^2+4x+1)^2}$$

$$y''>0 \iff \begin{cases} x<-1 \\ \dfrac{10x+4}{(5x^2+4x+1)^2}>0 \end{cases} \vee \begin{cases} -1<x<0 \\ -\dfrac{10x+4}{(5x^2+4x+1)^2}>0 \end{cases} \iff$$

$$\begin{cases} x<-1 \\ x>-\frac{2}{5} \end{cases} \vee \begin{cases} -1<x<0 \\ x<-\frac{2}{5} \end{cases} \iff -1<x<-\frac{2}{5}$$





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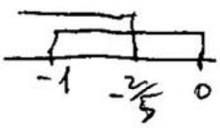

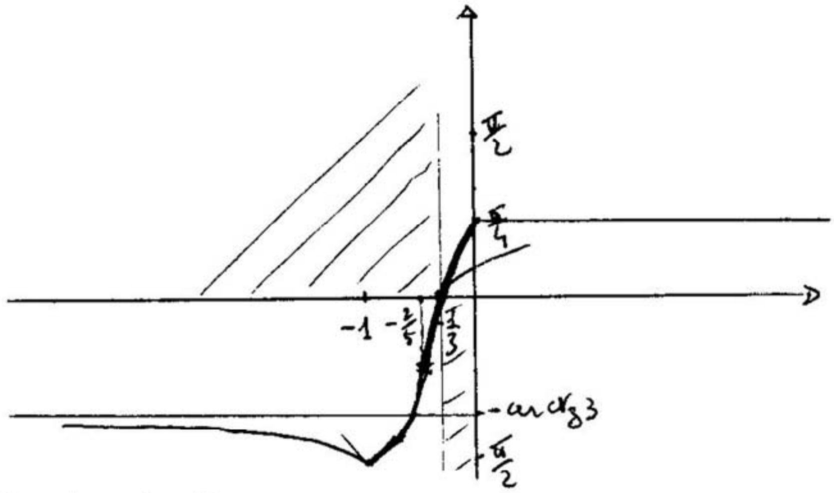

2. $f(x)=\dfrac{\lg(x^2+1)}{e^x-1}$ è continua per $x\neq 0$ e quindi in $]0,1]$ e in $[1,+\infty)$.

$$\lim_{x\to 0}\frac{\lg(x^2+1)}{e^x-1}=\lim_{x\to 0}\frac{x^{\cancel{2}}}{\cancel{x}}=0 \Rightarrow f \text{ integrabile in }]0,1]$$

$$\lim_{x\to+\infty}\frac{\lg(x^2+1)}{e^x-1}=0 \quad \text{con ordine molto grande e quindi} \geq \alpha>1$$

(perché $\lg(x^2+1)\to\infty$ con ord. comunque piccolo per $x\to+\infty$
$e^x-1\to\infty$ con ord. comunque grande per $x\to+\infty$)

e dunque f è integrabile in $[1,+\infty)$.

3. Poiché per $x\to 0$

$\sqrt{e^x-1}\simeq\sqrt{x}$, $\lg(\sqrt{x}+1)\simeq\sqrt{x}$ si ha $\sqrt{e^x-1}\cdot\lg(\sqrt{x}+1)\simeq\sqrt{x}\cdot\sqrt{x}=x$

$\operatorname{arctg}(\operatorname{sen}x)\simeq\operatorname{sen}x\simeq x$ e quindi il numeratore ha ord >1.

Il denominatore invece ha ordine 1 perché

$\operatorname{sen}(\lg(\operatorname{sen}x+1))\simeq\lg(\operatorname{sen}x+1)\sim\operatorname{sen}x$.

Si conclude perciò che il limite vale 0.

4. Posto $t = \operatorname{tg}\frac{x}{2} \Rightarrow x = 2\operatorname{arctg} t \Rightarrow dx = \frac{2}{1+t^2}\,dt$ l'integrale diventa

$$\int \frac{1}{\frac{1-t^2}{1+t^2} + 2\,\frac{2t}{1+t^2} + 1} \cdot \frac{2}{1+t^2}\,dt = \int \frac{2}{1-\cancel{t^2}+4t+1+\cancel{t^2}}\,dt = \int \frac{1}{2t+1}\,dt$$

$$= \frac{1}{2}\lg|2t+1| + c = \frac{1}{2}\lg\left|2\operatorname{tg}\frac{x}{2}+1\right| + c.$$

5. $\omega_{1,2} = i \pm \sqrt{i^2-i} = i + \sqrt{-1-i}$

Poiché $-1-i = \sqrt{2}\left(\cos\frac{5}{4}\pi + i\,\mathrm{sen}\frac{5}{4}\pi\right)$

$$\sqrt{-1-i} = \left\{\sqrt[4]{2}\left(\cos\frac{\frac{5}{4}\pi+2k\pi}{2} + i\,\mathrm{sen}\frac{\frac{5}{4}\pi+2k\pi}{2}\right) \,\middle|\, k=0,1\right\}$$

$$= \pm\sqrt[4]{2}\left(\cos\frac{5}{8}\pi + i\,\mathrm{sen}\frac{5}{8}\pi\right)$$

e quindi le soluzioni dell'equazione sono

$$\pm\sqrt[4]{2}\cos\frac{5}{8}\pi + i\left(1 \pm \sqrt[4]{2}\,\mathrm{sen}\frac{\pi}{8}\right).$$

Volendo si possono calcolare $\cos\frac{5}{8}\pi$ e $\mathrm{sen}\frac{5}{8}\pi$ con le formule di bisezione: $\cos^2\frac{\vartheta}{2} = \frac{1+\cos\vartheta}{2}$ $\quad \mathrm{sen}^2\frac{\vartheta}{2} = \frac{1-\cos\vartheta}{2}$.

Poiché $\frac{5}{8}\pi \in \left]\frac{\pi}{2},\pi\right[$: $\cos\frac{5}{8}\pi = -\sqrt{\frac{1+\cos\frac{5}{4}\pi}{2}} = -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{2}}{4}}$

$$\mathrm{sen}\frac{5}{8}\pi = \sqrt{\frac{1-\cos\frac{5}{4}\pi}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}}$$

e quindi

$$\omega_1 = \sqrt[4]{2}\left(-\frac{\sqrt{2-\sqrt{2}}}{2}\right) + i\left(1+\sqrt[4]{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\right)$$

$$\omega_2 = +\sqrt[4]{2}\cdot\frac{2-\sqrt{2}}{2} - i\left(1+\sqrt[4]{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\right)$$