COMPITO DI ISTITUZIONI DI MATEMATICHE I
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Esercizio 1. Studiare la funzione

$$f(x) = \log\left(\left|\frac{e^x - e^{-x}}{2} - 2\right|\right)$$

e disegnarne il grafico approssimativo.

Esercizio 2. Stabilire se la seguente funzione

$$f(x) = \frac{\sqrt{x}\,\mathrm{sen}\left(\frac{1}{x^2}\right)}{\log(1+x)}$$

è integrabile negli intervalli $]0,1]$, $[1,\infty[$.

Esercizio 3. Senza utilizzare il Teorema di de l'Hopital calcolare il seguente limite

$$\lim_{x\to 0} \frac{\log(1+\mathrm{sen}(x)) - \log((x+1)\cos(x))}{\mathrm{tang}^2(x) + e^{-\frac{1}{x}}}.$$

Esercizio 4. Calcolare il seguente integrale indefinito

$$\int (9x^2 + 6x)\log(1+x)dx.$$

Esercizio 5. Risolvere la seguente equazione in campo complesso

$$w^4 - 2\left(\cos\left(\frac{\pi}{8}\right) + i\,\mathrm{sen}\left(\frac{\pi}{8}\right)\right)\left(\cos\left(\frac{3\pi}{8}\right) + i\,\mathrm{sen}\left(\frac{3\pi}{8}\right)\right)w^2 +$$

$$+ i\left(\cos\left(\frac{5\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{5\pi}{4}\right)\right)\left(\cos\left(\frac{\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{\pi}{4}\right)\right) = 0.$$

1. ID $\left|\frac{e^x-e^{-x}}{2}-2\right|>0 \iff e^x-e^{-x}-4\neq 0 \iff e^{2x}-4e^x-1\neq 0$

$e^x = 2\pm\sqrt{5}$ $\qquad e^x=2-\sqrt{5}<0$ mai $\qquad e^x=2+\sqrt{5} \Rightarrow x=\lg(2+\sqrt{5})$

$ID = \mathbb{R}-\lg(2+\sqrt{5})$

$\begin{cases} x=0 \\ y=\lg 2 \end{cases} \qquad \begin{cases} y=0 \\ \lg\left|\frac{e^x-e^{-x}}{2}-2\right|=0 \end{cases} \Rightarrow \left|\frac{e^x-e^{-x}}{2}-2\right|=1 \Rightarrow$

$\frac{e^x-e^{-x}}{2}-2=1 \vee \frac{e^x-e^{-x}}{2}-2=-1 \Rightarrow e^{2x}-6e^x-1=0 \vee e^{2x}-2e^x-2=0$

$\Rightarrow e^x=3\pm\sqrt{10} < \begin{matrix} e^x=3-\sqrt{10} \ \emptyset \\ e^x=3+\sqrt{10} \quad x=\lg(3+\sqrt{10}) \end{matrix} \qquad \vee \qquad e^x=1\pm\sqrt{3} \Downarrow x=\lg(1+\sqrt{3})$

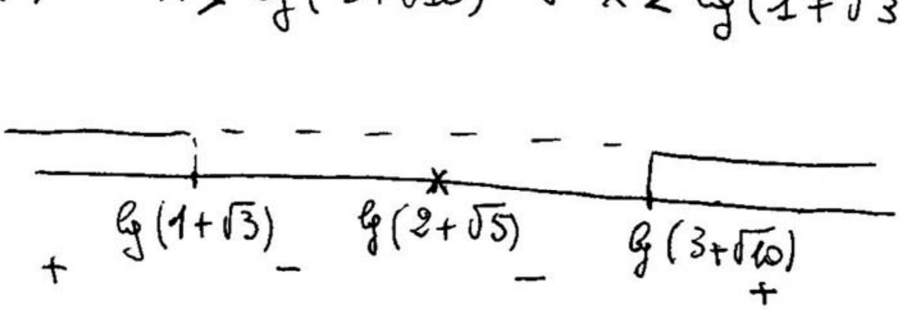
$(\lg(3+\sqrt{10}),0) \qquad (\lg(1+\sqrt{3}),0)$ intersezioni con asse x

$f(x)>0 \iff \left|\frac{e^x-e^{-x}}{2}-2\right|>1 \iff \frac{e^x-e^{-x}}{2}-2<-1 \vee \frac{e^x-e^{-x}}{2}-2>1$

$\iff e^x-e^{-x}-6>0 \vee e^x-e^{-x}-2<0$

$\iff \underset{<0}{e^x<3-\sqrt{10}} \vee e^x>3+\sqrt{10} \qquad \vee \qquad \underset{<0}{1-\sqrt{3}}<e^x<1+\sqrt{3}$

$\iff \quad x>\lg(3+\sqrt{10}) \vee x<\lg(1+\sqrt{3})$



Asintoti

$\lim_{x\to\pm\infty} \lg\left(\left|\frac{e^x-e^{-x}}{2}-2\right|\right)=+\infty \qquad \nexists$ as. orizzontali

$\lim_{x\to+\infty} \frac{\lg\left|\frac{e^x-e^{-x}}{2}-2\right|}{x} \overset{H}{=} \lim_{x\to+\infty} \frac{1}{\frac{e^x-e^{-x}}{2}-2}\cdot\frac{e^x+e^{-x}}{2}=$

$$= \lim_{x \to +\infty} \frac{e^x + e^{-x}}{e^x - e^{-x} - 4} = \lim_{x \to +\infty} \frac{e^x}{e^x} = 1$$

$$\lim_{x \to +\infty} f(x) - x = \lim_{x \to +\infty} \lg \left| \frac{e^x - e^{-x}}{2} - 2 \right| - x = \lim_{x \to +\infty} \lg \left| \frac{e^x - e^{-x}}{2} - 2 \right| - \lg e^x$$

$$= \lim_{x \to +\infty} \lg \left| \frac{\frac{e^x - e^{-x}}{2} - 2}{e^x} \right| = \lim_{x \to +\infty} \lg \left| \frac{e^x - e^{-x} - 4}{2e^x} \right| = \lim_{x \to +\infty} \lg \left| \frac{\cancel{e^x}}{2\cancel{e^x}} \right| =$$

$$= \lg \frac{1}{2} = -\lg 2 \qquad y = x - \lg 2 \quad \text{As. ob. a } +\infty$$

$$\lim_{x \to -\infty} \frac{f(x)}{x} \overset{H}{=} \lim_{x \to -\infty} \frac{e^x + e^{-x}}{e^x - e^{-x} - 4} = \lim_{x \to +\infty} \frac{e^{-x}}{-e^{-x}} = -1$$

$$\lim_{x \to -\infty} \lg \left| \frac{e^x - e^{-x}}{2} - 2 \right| + x = \lim_{x \to -\infty} \lg \left| \frac{e^x - e^{-x}}{2} - 2 \right| + \lg e^x$$

$$= \lim_{x \to -\infty} \lg \left| e^x \left(\frac{e^x - e^{-x}}{2} - 2 \right) \right| = \lim_{x \to -\infty} \lg \left| \frac{e^{2x} - 1 - 4e^x}{2} \right| = \lg \frac{1}{2}$$

$$y = -x - \lg 2 \qquad \text{As. ob. a } -\infty$$

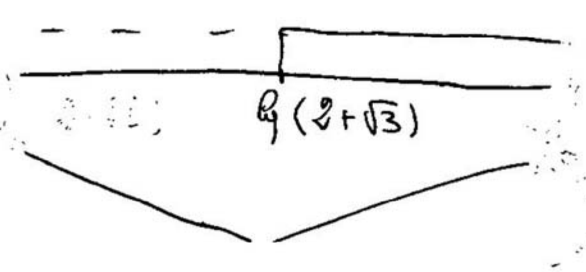
$$\lim_{x \to \lg(2+\sqrt{5})} f(x) = \lg |0| = -\infty \qquad x = \lg(2+\sqrt{5}) \quad \text{As. obliquo}$$

Studio della derivata 1ª

$$y' = \frac{1}{\frac{e^x - e^{-x}}{2} - 2} \left(\frac{e^x + e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{e^x - e^{-x} - 4} \qquad ID(y') = ID(y)$$

$$y' > 0 \iff e^x - e^{-x} - 4 > 0 \iff e^{2x} - 4e^x - 1 > 0 \iff$$

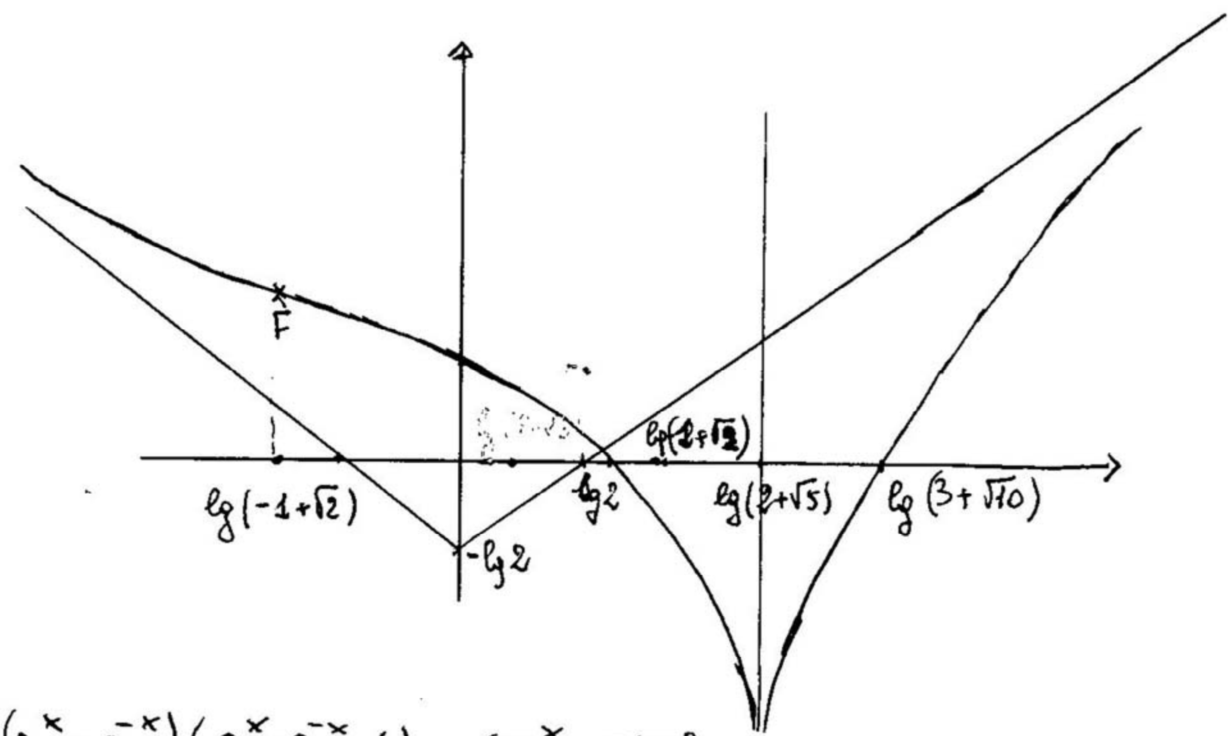
$$e^x < \underset{<0}{2 - \sqrt{3}} \ \vee \ e^x > 2 + \sqrt{3} \iff x > \lg(2+\sqrt{3})$$



f str. decrescente in $(-\infty, \lg(2+\sqrt{5})[$

f str. crescente in $]\lg(2+\sqrt{5}), +\infty)$

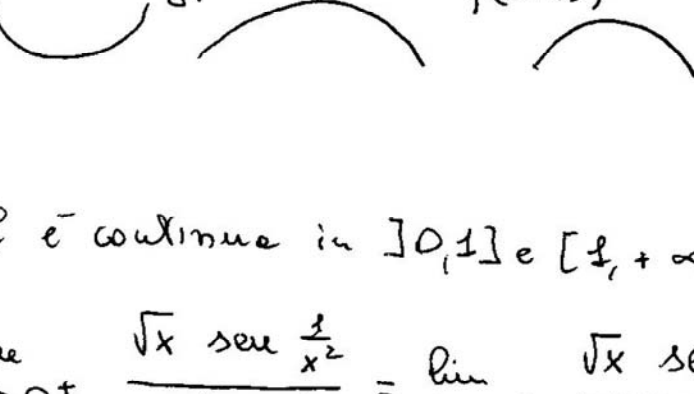
$\nexists$ estremi relativi



$$y'' = \frac{(e^x - e^{-x})(e^x - e^{-x} - 4) - (e^x - e^{-x})^2}{(e^x - e^{-x} - 4)^2} = \frac{-4e^x + 4e^{-x} - 4}{(e^x - e^{-x} - 4)^2}$$

$$y'' > 0 \iff -e^x + e^{-x} - 1 > 0 \iff e^{2x} + e^x - 1 < 0$$

$$\underset{<0}{-1-\sqrt{2}} < e^x < -1 + \sqrt{2} \iff x < \lg(-1+\sqrt{2})$$



f convessa in $(-\infty, \lg(-1+\sqrt{2})[$

f concava in $]\lg(-1+\sqrt{2}), \lg(2+\sqrt{5})[$

e in $]\lg(2+\sqrt{5}), +\infty)$

f ha un flesso in $x = \lg(-1+\sqrt{2})$.

2. f è continua in $]0,1]$ e $[1,+\infty)$. Inoltre

$$\lim_{x\to 0^+} \frac{\sqrt{x}\,\operatorname{sen}\frac{1}{x^2}}{\lg(1+x)} \underset{\uparrow}{=} \lim_{x\to 0^+} \frac{\sqrt{x}\,\operatorname{sen}\frac{1}{x^2}}{x} \underset{\uparrow}{=} \lim_{y\to+\infty} \sqrt{y}\,\operatorname{sen} y^2 \quad \nexists$$

$\lg(1+x) \sim x$ se $x \to 0^+$ $\qquad y = \frac{1}{x}$

perché se mi restringo a $\{2k\pi \mid k \in \mathbb{Z}\}$ il limite vale 0

" $\{\frac{\pi}{2} + 2k\pi \mid k \in \mathbb{Z}\}$ " " $+\infty$.

Poiché però $0 \le f(x) \le \frac{\sqrt{x}}{\lg(1+x)}$ $\forall x \in]0,1]$ e $\lim_{x\to 0^+} \frac{\sqrt{x}}{\lg(1+x)} = +\infty$ con ordine $\frac{1}{2} < 1$, per confronto f è integrabile in $]0,1]$.

3. Poiché ha senso calcolare il limite sia da destra che da sinistra, distinguo i casi:

$\lim_{x\to 0^-} f(x) = \frac{0}{+\infty} = 0$ non c'è forma indeterminata.

$$\lim_{x\to 0^+} f(x) = \frac{0}{0} = \lim_{x\to 0^+} \frac{\lg \frac{1+\operatorname{sen} x}{(x+1)\cos x}}{\operatorname{tg}^2 x + e^{-\frac{1}{x}}} = \lim_{x\to 0^+} \frac{\lg\left(1+\frac{1+\operatorname{sen} x}{(x+1)\cos x}-1\right)}{\operatorname{tg}^2 x + \cancel{e^{-\frac{1}{x}}}}$$

$\lg(1+y) \sim y$ se $y \to 0$

$\operatorname{ord} \operatorname{tg}^2 x = 2 < \operatorname{ord}(e^{-\frac{1}{x}})$

$$= \lim_{x\to 0^+} \frac{\frac{1+\operatorname{sen} x - \cancel{\cos x} - x\cos x}{(x+1)\cos x}}{x^2} =$$

$1-\cos x \cong \frac{1}{2}x^2 - \frac{1}{12}x^4$; $\operatorname{sen} x \cong x - \frac{x^3}{6}$

$$= \lim_{x\to 0^+} \underbrace{\frac{1}{(x+1)\cos x}}_{\to 1} \cdot \frac{\frac{1}{2}x^2 + x - \frac{x^3}{6} - x\left(1-\frac{1}{2}x^2+\cancel{\frac{1}{12}x^4}\right)}{x^2} =$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{2}x^2 + \cancel{x} - \frac{x^3}{6} - \cancel{x} + \frac{1}{2}x^3}{x^2} = \frac{1}{2}$$

4. $\int (9x^2+6x)\lg(1+x)\,dx = \int D\left(\cancel{9}^{3}\frac{x^3}{\cancel{3}} + \cancel{6}^{3}\frac{x^2}{\cancel{2}}\right)\lg(1+x)\,dx$

$$= 3(x^3+x^2)\lg(1+x) - 3\int \frac{x^3+x^2}{1+x}\,dx = 3(x^3+x^2)\lg(1+x) - x^3 + c$$

5. Poiché $2\left(\cos\frac{\pi}{8} + i\operatorname{sen}\frac{\pi}{8}\right)\left(\cos\frac{3}{8}\pi + i\operatorname{sen}\frac{3}{8}\pi\right) = 2\left(\cos\frac{\pi}{2} + i\operatorname{sen}\frac{\pi}{2}\right) = 2i$

$\left(\cos\frac{5}{4}\pi + i\operatorname{sen}\frac{5}{4}\pi\right)\left(\cos\frac{\pi}{4} + i\operatorname{sen}\frac{\pi}{4}\right) = \cos\frac{3}{2}\pi + i\operatorname{sen}\frac{3}{2}\pi = -i$

l'equazione diventa $\omega^4 - 2i\omega^2 + 1 = 0 \Rightarrow$

$\omega^2 = i + \sqrt{i^2-1} = i + \sqrt{-2} = (1 \pm \sqrt{2})i$

$$\omega_{1,2} = \sqrt{(1+\sqrt{2})i} = \left\{\sqrt{1+\sqrt{2}}\left(\cos\frac{\frac{\pi}{2}+2k\pi}{2} + i\operatorname{sen}\frac{\frac{\pi}{2}+2k\pi}{2}\right) \mid k=0,1\right\}$$

$$= \left\{\sqrt{1+\sqrt{2}}\left(\cos\frac{\pi}{4} + i\operatorname{sen}\frac{\pi}{4}\right),\ \sqrt{1+\sqrt{2}}\left(\cos\frac{5}{4}\pi + i\operatorname{sen}\frac{5}{4}\pi\right)\right\} = \pm\sqrt{\frac{\sqrt{2}+2}{2}}(1+i)$$

$$\omega_{3,4} = \sqrt{(1-\sqrt{2})i} = \left\{\sqrt{\sqrt{2}-1}\left(\cos\frac{\frac{3}{2}\pi+2k\pi}{2} + i\operatorname{sen}\frac{\frac{3}{2}\pi+2k\pi}{2}\right) \mid k=0,1\right\}$$

$$= \left\{\pm\sqrt{\sqrt{2}-1}\left(\cos\frac{3}{4}\pi + i\operatorname{sen}\frac{3}{4}\pi\right)\right\} = \pm\sqrt{\frac{2-\sqrt{2}}{2}}(-1+i)$$