Corso di laurea in CHIMICA
Esame di
ISTITUZIONI di MATEMATICHE I

27 aprile 2011

1. Studiare la funzione
$$f(x) = \lg_{0,5}\left(1 - \left|\frac{x}{2x-1}\right|\right)$$
e disegnarne il grafico approssimativo.

2. Risolvere la seguente equazione nel campo complesso
$$|z|^4 z = \frac{\bar{z}}{|z|^2}.$$
e disegnarne se possibile le soluzioni nel piano complesso.

3. Calcolare il seguente limite (senza usare la regola di de L'Hopital)
$$\lim_{x\to 0} \frac{\tan(\cos x - 1) + \frac{1}{2}\arctan x^2 - \frac{1}{24}x^4}{\sin x^2 + \tan x^2}.$$

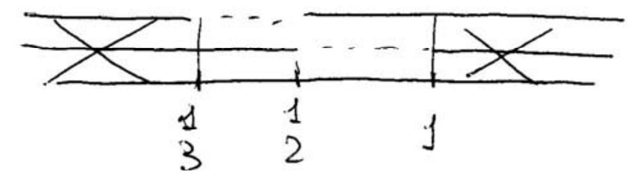
4. Studiare il dominio di
$$y = \frac{\left(3^{2|x|} - 3^{x+1} - 4\right)^{\sqrt{x}} + \sqrt{\arctan x}}{(\ln x^2)^2 - \ln(ex)}.$$

5. Dopo aver studiato al variare di $\alpha \in \mathbf{R}$ la convergenza del seguente integrale improprio
$$\int_4^{+\infty} \frac{1}{x^\alpha\sqrt{x^2-16}},$$
calcolarne il valore per $\alpha = 1$.

1. $f(x) = \lg_{0,5}\left(1 - \left|\frac{x}{2x-1}\right|\right)$

ID: $1 - \left|\frac{x}{2x-1}\right| > 0 \iff \left|\frac{x}{2x-1}\right| < 1 \iff \begin{cases} \frac{x-2x+1}{2x-1} < 0 \\ \frac{x+2x-1}{2x-1} > 0 \end{cases} \iff \begin{cases} \frac{x-1}{2x-1} > 0 \\ \frac{3x-1}{2x-1} > 0 \end{cases}$

$\begin{cases} x < \frac{1}{2} \vee x > 1 \\ x < \frac{1}{3} \vee x > \frac{1}{2} \end{cases}$



$ID = (-\infty, \frac{1}{3}[\cup]1, +\infty)$

$f(x) > 0 \iff 0 < 1 - \left|\frac{x}{2x-1}\right| < 1 \iff \begin{cases} \left|\frac{x}{2x-1}\right| > 0 \\ x \in ID \end{cases}$ $\forall x \in ID \quad x \neq 0$

$f(x) = 0 \iff \cancel{1} - \left|\frac{x}{2x-1}\right| = \cancel{1} \iff \frac{x}{2x-1} = 0 \iff x = 0$ $(0,0)$ unica intersezione con gli assi

$\lim_{x \to \frac{1}{3}^-} f(x) = \lim_{y \to 0^+} \lg_{0,5} y = +\infty$

$\lim_{x \to 1^+} f(x) = \lim_{y \to 0^+} \lg_{0,5} y = +\infty$

$\Rightarrow$ $x = \frac{1}{3}$ As. vert.

$x = 1$ As. vert.

$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \lg_{0,5}\left(1 - \left|\frac{\cancel{x}}{\left(2 - \frac{1}{x}\right)\cancel{x}}\right|\right) = \lg_{0,5} \frac{1}{2} = 1$

$y = 1$ As. orizz. a $\pm\infty$

$$y' = \frac{1}{1 - \left|\frac{x}{2x-1}\right|} \cdot \frac{1}{\lg 0,5}\left(-\frac{\frac{x}{2x-1}}{\left|\frac{x}{2x-1}\right|}\right)\frac{\cancel{2x} - 1 - \cancel{2x}}{(2x-1)^2} = \frac{1}{1 - \left|\frac{x}{2x-1}\right|} \frac{1}{\lg 0,5} \cdot \frac{\frac{x}{2x-1}}{\left|\frac{x}{2x-1}\right|} \cdot \frac{1}{(2x-1)^2}$$

$ID(y'): \begin{cases} \frac{x}{2x-1} \neq 0 \\ x \in ID \end{cases} \Rightarrow ID(y') = ID(y) - \{0\}$

$\lim_{x \to 0^-} y' = \lim_{x \to 0^-} \frac{1}{1 - \left|\frac{x}{2x-1}\right|} \frac{1}{\lg 0,5} \frac{1}{(2x-1)^2} = \frac{1}{\lg 0,5} = -\frac{1}{\lg 2}$

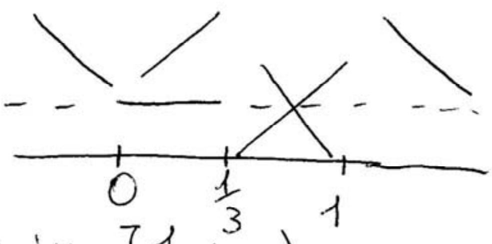
$\lim_{x \to 0^+} y' = \lim_{x \to 0^+} \frac{1}{1 - \left|\frac{x}{2x-1}\right|} \cdot \frac{1}{\lg 0,5} \frac{1}{(2x-1)^2} = +\frac{1}{\lg 2}$

$x = 0$ pto angoloso

$$y'>0 \iff \underbrace{\frac{1}{1-\left|\frac{x}{2x-1}\right|}}_{\overset{\vee}{0} \text{ in ID}}\cdot\underbrace{\frac{1}{\lg 0{,}5}}_{\overset{\wedge}{0}}\;\frac{\frac{x}{2x-1}}{\left|\frac{x}{2x-1}\right|}\cdot\frac{1}{(2x-1)^2}>0 \iff \frac{x}{2x-1}<0$$

$$\iff \begin{cases} 0<x<\frac{1}{2} \\ x\in ID \end{cases}$$

$y'>0 \iff 0<x<\frac{1}{3}$



f decresce in $(-\infty,0[$ e in $]1,+\infty)$

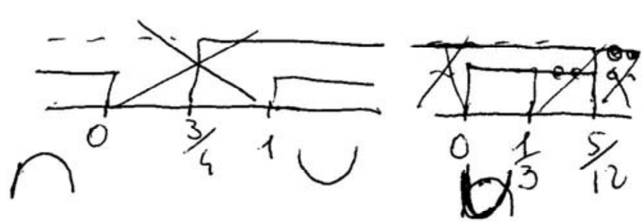
f cresce in $]0,\frac{1}{3}[$ $\qquad x=0$ pto di min rel e assoluto

$$y''=\begin{cases} D\,\frac{1}{1-\frac{x}{2x-1}}\cdot\frac{1}{-\lg 2}\cdot\frac{1}{(2x-1)^2}=D\,\frac{-1}{(x-1)\ln 2\,(2x-1)} & x<0 \vee x>1 \\ D\,\frac{1}{1+\frac{x}{2x-1}}\cdot\frac{-1}{-\lg 2}\,\frac{1}{(2x-1)^2}=D\,\frac{1}{\lg 2\,(2x-1)(3x-1)} & 0<x<\frac{1}{3} \end{cases}$$

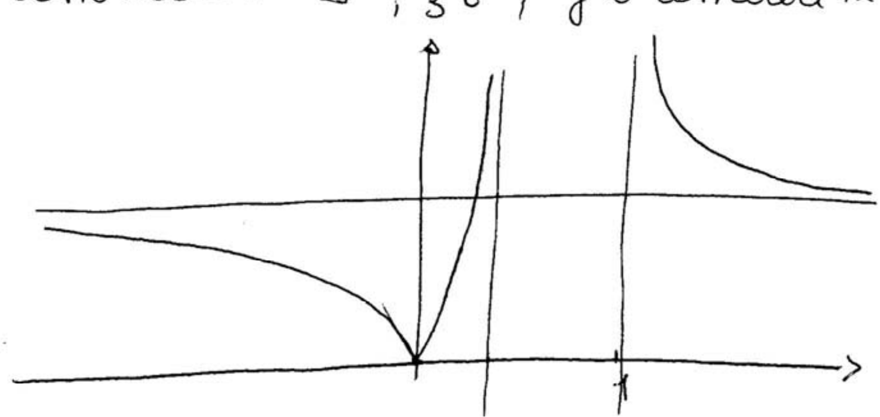
$$y''=\begin{cases} \frac{1}{\lg 2}\,\frac{2x-1+2(x-1)}{(x-1)^2(2x-1)^2}=\frac{1}{\lg 2}\,\frac{4x-3}{(x-1)^2(2x-1)^2} & x<0 \vee x>1 \\ -\frac{1}{\lg 2}\,\frac{2(3x-1)+3(2x-1)}{(2x-1)^2(3x-1)^2}=-\frac{1}{\lg 2}\,\frac{12x-5}{(2x-1)^2(3x-1)^2} & 0<x<\frac{1}{3} \end{cases}$$

$$y''>0 \iff \begin{cases} x<0 \vee x>1 \\ 4x-3>0 \end{cases} \vee \begin{cases} 0<x<\frac{1}{3} \\ -12x+5>0 \end{cases} \iff$$

$$\begin{cases} x<0 \vee x>1 \\ x>\frac{3}{4} \end{cases} \vee \begin{cases} 0<x<\frac{1}{3} \\ x<\frac{5}{12} \end{cases}$$



f è convessa in $]0,\frac{1}{3}[$, f è concava in $]-\infty,0[$ e in $]1,+\infty)$.



2. $|z|^4 \cdot z = \frac{\bar z}{|z|^2} \iff \begin{cases}|z|^6 \cdot z = \bar z \\ z \neq 0\end{cases} \iff \begin{cases}|z|^6 z \cdot \bar z = (\bar z)^2 \\ z \neq 0\end{cases} \iff \begin{cases}|z|^8 = (\bar z)^2 \\ z \neq 0\end{cases}$

poiché $|z|^2 = x^2+y^2$

$(\bar z)^2 = (x-iy)^2 = x^2-y^2-2xyi$ segue che

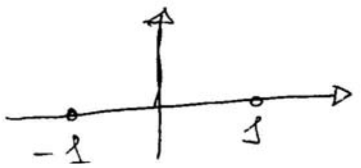
$\begin{cases}(x^2+y^2)^{\cancel{4}} = x^2 - y^2 \\ xy = 0\end{cases} \iff \begin{cases}x = 0 \\ y^8 = -y^2 \Rightarrow y = 0\end{cases}$ $\vee$ $\begin{cases}y = 0 \\ x^8 = x^2\end{cases}$

$z = 0$ Non accett. $\quad x^2(x^6-1) = 0$

$\begin{cases}x = 0 \ \vee\ x^6 = 1 \Rightarrow x = \pm 1 \\ y = 0\end{cases}$

Non acc.

Le uniche sol. dell'equazione sono i numeri $z = \pm 1$



3. Poiché $tg(\cos x - 1) \sim \cos x - 1 \sim -\frac{1}{2}x^2$ per $x \to 0$

$arctg\, x^2 \sim x^2 \quad sen\, x^2 \sim x^2 \quad tg\, x^2 \sim x^2$

il limite diventa

$$\lim_{x \to 0} \frac{-\cancel{\frac{1}{2}x^2} + \cancel{\frac{1}{2}x^2} - \frac{1}{24}x^4}{x^2 + x^2} = \lim_{x \to 0} \frac{-\frac{1}{24}x^{\cancel{4}\,2}}{2\cancel{x^2}} = 0$$

4. $\begin{cases}3^{2|x|} - 3^{x+1} - 4 \geq 0 \\ x \geq 0 \\ arctg\, x \geq 0 \\ x^2 > 0 \Rightarrow x \neq 0 \\ (lg\, x^2)^2 - lg(ex) \neq 0\end{cases} \qquad \begin{cases}3^{2|x|} - 3 \cdot 3^x - 4 \geq 0 \\ x > 0 \\ (2\, lg|x|)^2 - lg\, e - lg\, x \neq 0\end{cases}$

Poiché $x > 0$, la 1ª disep. diventa $3^{2x} - 3 \cdot 3^x - 4 \geq 0$

$3^x_{1,2} = \frac{3 \pm 5}{2} = \begin{cases}-1 \\ 4\end{cases} \qquad 3^x = -1$ imp. $\vee$ $3^x = 4 \quad x = lg_3 4$

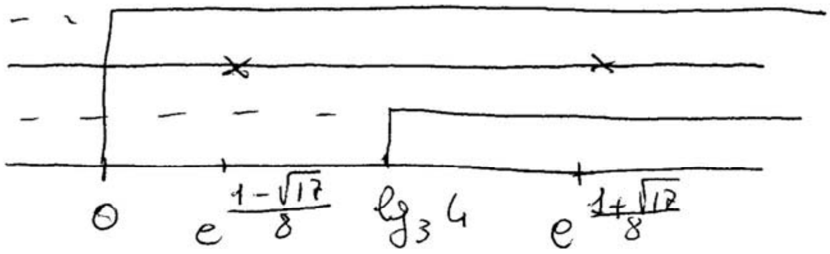
$3^x \leq -1 \quad \vee \quad 3^x \geq 4 \quad x \geq lg_3 4$

$\emptyset$

Risolvendo l'ultima relazione si ha $4\, lg^2 x - lg\, x - 1 \neq 0$

$$\lg x \neq \frac{1 \pm \sqrt{17}}{8} \Rightarrow x \neq e^{\frac{1\pm\sqrt{17}}{8}}$$

$$\begin{cases} x \geq \lg_3 4 \\ x > 0 \\ x \neq \cancel{\lg}\ e^{\frac{1\pm\sqrt{17}}{8}} \end{cases}$$



$$ID =]\lg_3 4, +\infty) - \left\{e^{\frac{1+\sqrt{17}}{8}}\right\}$$

5. Poiché $\lim_{x\to 4^+} \frac{1}{x^\alpha\sqrt{x^2-16}} = \lim_{x\to 4^+} \frac{1}{x^\alpha\sqrt{x+4}\sqrt{x-4}} = +\infty$ con ord $\frac{1}{2} < 1$

la funzione è integrabile in senso improprio in $]4, 5]$.

Poiché $\lim_{x\to+\infty} \frac{1}{x^\alpha\sqrt{x^2-16}} = \lim_{x\to+\infty} \frac{1}{x^\alpha \cdot x} = \begin{cases} 0 & \alpha+1>0 \\ 1 & \alpha+1=0 \\ +\infty & \alpha+1<0 \end{cases}$

f è integrabile in senso improprio in $[5, +\infty)$ se e solo se $\alpha + 1 > 1 \iff \alpha > 0$. Si conclude che $\int_4^{+\infty} f(x)\,dx$ converge $\iff \alpha > 0$

Calcoliamone il valore per $\alpha = 1$

Posto $\sqrt{x^2-16} = x - t$ si ha $x = \frac{t^2+16}{2t} = \frac{1}{2}\left(t + \frac{16}{t}\right)$ $dx = \frac{1}{2}\left(1 - \frac{16}{t^2}\right)dt$

$$\int \frac{1}{x\sqrt{x^2-16}}\,dx = \int \frac{1}{\frac{t^2+16}{2t}\cdot\frac{16-t^2}{2t}}\cdot\frac{1}{2}\left(\frac{t^2-16}{t^2}\right)dt \qquad \sqrt{x^2-16} = \frac{t^2+16}{2t} - t = \frac{16-t^2}{2t}$$

$$= -\int \frac{2}{t^2+16}\,dt = -2\int \frac{1}{16\left(\left(\frac{t}{4}\right)^2+1\right)}dt = -\frac{2}{4}\int \frac{1/4}{\left(\frac{t}{4}\right)^2+1}dt = -\frac{1}{2}\operatorname{arctg}\frac{t}{4}$$

$$\int_4^{+\infty} \frac{1}{x\sqrt{x^2-16}}\,dx = \int_4^{+\infty} -\frac{1}{2}\operatorname{arctg}\frac{x-\sqrt{x^2-16}}{4}\,dx = -\frac{1}{2}\left(\lim_{x\to+\infty}\operatorname{arctg}\frac{x-\sqrt{\ }}{4} - \frac{\pi}{4}\right)$$

$$= -\frac{1}{2}\left(\operatorname{arctg} 0 - \frac{\pi}{4}\right) = \frac{\pi}{8}.$$

poiché $\lim_{x\to+\infty}\left(x - \sqrt{x^2-16}\right) = \lim_{x\to+\infty}\frac{\cancel{x^2}-\cancel{x^2}+16}{x+\sqrt{x^2-16}} = 0$

N.B. Si può anche usare la sostituzione $\sqrt{x^2-16} = t$ che in questo caso funziona, anche se il radicando è un polinomio di II gr.