COMPITO DI ISTITUZIONI DI MATEMATICHE I
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Esercizio 1. Studiare la funzione

$$f(x) = x\left|2 + \frac{1}{\log(x)}\right|$$

e disegnarne il grafico approssimativo.

Esercizio 2. Dire, giustificando le risposte, se la funzione

$$f(x) = \frac{1-\cos(x)}{\sqrt[4]{|1-x^3|^3}}$$

è integrabile negli intervalli $[0,1[$, $]1,+\infty[$.

Esercizio 3. Utilizzando il Teorema di de l'Hopital calcolare il seguente limite

$$\lim_{x\to\frac{\pi}{4}} \frac{e^{\operatorname{sen}(x)} - e^{\cos(x)} + \log\left(\left(x-\frac{\pi}{4}\right)^2+1\right)}{\operatorname{arctang}(\operatorname{sen}(x)) - \operatorname{arctang}(\cos(x))}.$$

Esercizio 4. Confrontando gli infinitesimi calcolare il seguente limite

$$\lim_{x\to 0} \frac{\cos(e^{x^2}-1) - e^{x^3}}{\log(\operatorname{arctang}(x)+1)}.$$

Esercizio 5. Data la funzione

$$f(x) = \operatorname{sen}(e^{x\cos(x)}\operatorname{arctang}(\log(x))),$$

scrivere l'equazione della retta tangente al grafico di f nel punto di ascissa $x_0 = 1$.

Esercizio 6. Risolvere la seguente equazione in campo complesso

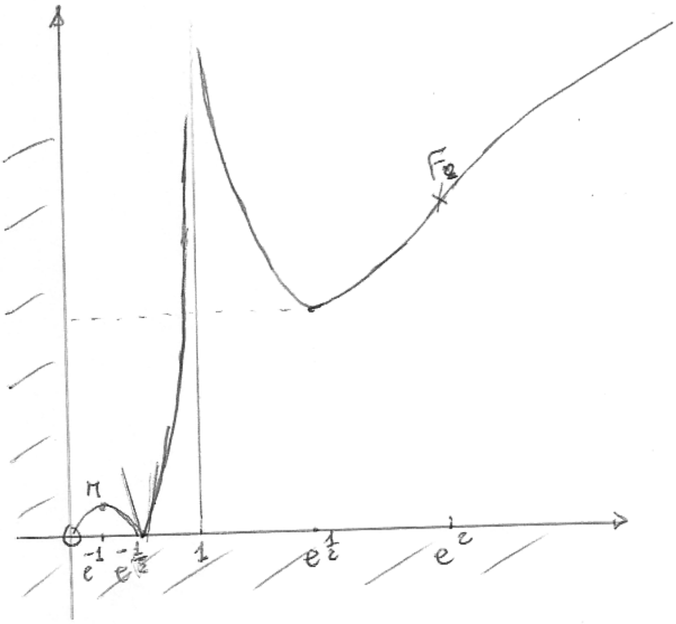
$$\overline{w}^5 = -\frac{2}{1+i} - 10\frac{2+i}{2-i}.$$

1. $\lg x \neq 0 \iff x > 0, x \neq 1 \qquad ID =]0,1[\cup]1,+\infty)$

Asintoti verticali

$\lim\limits_{x\to 0^+} x\left|2+\frac{1}{\lg x}\right| = 0$ f si può prolungare per continuità in $x=0$

$\lim\limits_{x\to 1} x\left|2+\frac{1}{\lg x}\right| = +\infty$ $x=1$ As. verticale completo



$f(x) \geq 0 \quad \forall x \in ID$

$f(x) = 0 \iff \lg x = -\frac{1}{2} \iff x = e^{-\frac{1}{2}}$

$\lim\limits_{x\to+\infty} x\left|2+\frac{1}{\lg x}\right| = +\infty$ $\nexists$ as. orizz.

$\lim\limits_{x\to+\infty} \frac{\cancel{x}\left|2+\frac{1}{\lg x}\right|}{\cancel{x}} = 2$ (eventuale coeff. angolare)

$\lim\limits_{x\to+\infty} x\left|2+\frac{1}{\lg x}\right| - 2x \overset{*}{=}$

$\lim\limits_{x\to+\infty} x\left(\cancel{2}+\frac{1}{\lg x}\right) - \cancel{2x} = \lim\limits_{x\to+\infty} \frac{x}{\lg x} = +\infty$

(per gli ordini) $\nexists$ as. obliqui

* Si può togliere il valore assoluto essendo $2+\frac{1}{\lg x} > 0 \iff \lg x < -\frac{1}{2} \vee \lg x > 0$ $\iff 0 < x < e^{-\frac{1}{2}} \vee x > 1$ (e quindi sicuramente l'argomento del valore assoluto è positivo per $x \to +\infty$).

$$f'(x) = \left|2+\frac{1}{\lg x}\right| + \cancel{x}\,\frac{2+\frac{1}{\lg x}}{\left|2+\frac{1}{\lg x}\right|}\cdot\frac{-1}{\cancel{x}\lg^2 x} = \frac{\left(2+\frac{1}{\lg x}\right)^2\lg^2 x - \left(2+\frac{1}{\lg x}\right)}{\lg^2 x\left|2+\frac{1}{\lg x}\right|}$$

$$= \frac{\left(2+\frac{1}{\lg x}\right)\left[\left(2+\frac{1}{\lg x}\right)\lg^2 x - 1\right]}{\lg^2 x\left|2+\frac{1}{\lg x}\right|} = \frac{(2\lg x+1)(2\lg^2 x+\lg x-1)}{\lg^3 x\left|2+\frac{1}{\lg x}\right|}$$

$$ID(f') = \left\{x \in ID(f) \,\middle|\, 2+\frac{1}{\lg x} \neq 0\right\} =]0, e^{-\frac{1}{2}}[\cup]e^{-\frac{1}{2}}, 1[\cup]1, +\infty).$$

$$\lim_{x\to(e^{-\frac{1}{2}})^\pm} y' = \lim_{x\to(e^{-\frac{1}{2}})^\pm} \left|2+\frac{1}{\lg x}\right| - \frac{2+\frac{1}{\lg x}}{\left|2+\frac{1}{\lg x}\right|}\cdot\frac{1}{\lg^2 x} = \mp\frac{1}{(\lg e^{-\frac{1}{2}})^2} = \mp 4$$

$\begin{cases} x = e^{-\frac{1}{2}} \\ y = 0 \end{cases}$ pto angoloso perché $f'_s(e^{-\frac{1}{2}}) = -4$, $f'_d(e^{-\frac{1}{2}}) = +4$.

$f'(x)=0 \iff 2\lg^2 x+\lg x-1=0 \iff \lg x=-1 \lor \lg x=\frac{1}{2} \iff x=\frac{1}{e} \lor x=\sqrt{e}$

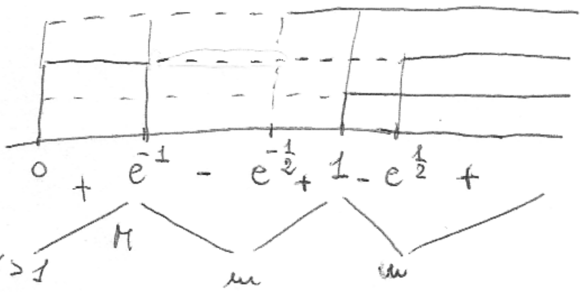
$f'(x)>0 \iff \dfrac{(2\lg x+1)(2\lg^2 x+\lg x-1)}{\lg^3 x}>0$

$\lg x>0 \qquad x>1$

$2\lg^2 x+\lg x-1>0 \iff \lg x<-1 \lor \lg x>\frac{1}{2} \iff 0<x<\frac{1}{e} \lor x>\sqrt{e}$

$\lg x>-\frac{1}{2} \qquad x>e^{-\frac{1}{2}}$

$M\begin{cases}x=e^{-1}\\ y=e^{-1}\end{cases} \qquad m\begin{cases}x=e^{-\frac{1}{2}}\\ y=0\end{cases} \qquad m\begin{cases}x=e^{\frac{1}{2}}\\ y=4e^{\frac{1}{2}}\end{cases}$



$y''=\begin{cases}D\left(2+\frac{1}{\lg x}-\frac{1}{\lg^2 x}\right) & 0<x<e^{-\frac{1}{2}} \lor x>1\\ D\left(-2-\frac{1}{\lg x}+\frac{1}{\lg^2 x}\right) & e^{-\frac{1}{2}}<x<1\end{cases}$

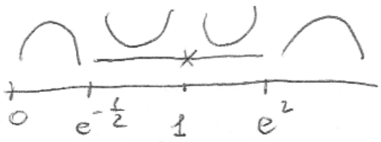
$y''>0 \iff \begin{cases}0<x<e^{-\frac{1}{2}} \lor x>1\\ -\frac{1}{x\lg^2 x}+\frac{2}{x\lg^3 x}>0\end{cases} \lor \begin{cases}e^{-\frac{1}{2}}<x<1\\ +\frac{1}{x\lg^2 x}-\frac{2}{x\lg^3 x}>0\end{cases}$

$\dfrac{-\lg x+2}{\lg^3 x}>0 \iff 0<\lg x<2 \iff 1<x<e^2$

$y''>0 \iff e^{-\frac{1}{2}}<x<1 \lor 1<x<e^2$

f convessa in $]e^{-\frac{1}{2}},1[$ e in $]1,e^2[$

f concava in $]0,e^{-\frac{1}{2}}[$ e in $]e^2,+\infty)$



$F_1\begin{cases}x=e^{-\frac{1}{2}}\\ y=0\end{cases} \qquad F_2\begin{cases}x=e^2\\ y=\frac{5}{2}e^2\end{cases} \qquad F_1$ e F_2 flessi

2. Poiché $\lim_{x\to 1^-} \frac{1-\cos x}{\sqrt[4]{|1-x^3|^3}} = \lim_{x\to 1^-} \frac{\overbrace{1-\cos x}^{\to 1-\cos 1>0}}{(1-x)^{3/4}\underbrace{(1+x+x^2)^{3/4}}_{3^{3/4}}} = +\infty$ con ord $\frac{3}{4}<1$

f è integrabile in s. i. in $[0,1[$.

Poiché $\lim_{x\to 1^+} \frac{1-\cos x}{(x-1)^{3/4}(x^2+x+1)^{3/4}} = +\infty$ con ord. $\frac{3}{4}<1$

mentre $\lim_{x\to +\infty} \frac{1-\cos x}{(x^3-1)^{3/4}} = 0$ con ord $\frac{9}{4}>1$

f è integrabile in $]1,+\infty[$.

3. $\lim_{x\to\frac{\pi}{4}} \frac{e^{\operatorname{sen} x} - e^{\cos x} + \lg\left(\left(x-\frac{\pi}{4}\right)^2+1\right)}{\operatorname{arctg} \operatorname{sen} x - \operatorname{arctg} \cos x} = \frac{0}{0} \overset{H}{=}$

$$\lim_{x\to\frac{\pi}{4}} \frac{e^{\operatorname{sen} x}\cdot\cos x + e^{\cos x}\cdot\operatorname{sen} x + \frac{2\left(x-\frac{\pi}{4}\right)}{\left(x-\frac{\pi}{4}\right)^2+1}}{\frac{1}{1+\operatorname{sen}^2 x}\cos x + \frac{1}{1+\cos^2 x}\cdot\operatorname{sen} x} = \frac{\frac{\sqrt{2}}{2}\cdot 2e^{\frac{\sqrt{2}}{2}}}{\frac{\sqrt{2}}{2}\cdot 2\cdot\frac{1}{1+\frac{1}{2}}} = \frac{3}{2}e^{\frac{\sqrt{2}}{2}}$$

4. Poiché $\cos x = 1-\frac{x^2}{2}+o(x^3)$ $\qquad e^x = 1+x+\frac{x^2}{2}+o(x^2)$

$\lg(1+x)\sim x$ $\qquad \lg(1+\operatorname{arctg} x)\sim \operatorname{arctg} x \sim x$

segue che $\cos(e^{x^2}-1) \sim 1-(e^{x^2}-1)^2 \sim 1-x^4$

$e^{x^3}\sim 1+x^3$

quindi il limite diventa

$$\lim_{x\to 0} \frac{\cancel{1}-x^4-\cancel{1}-x^3}{x} = \lim_{x\to 0} \frac{-x^3}{x} = 0$$

5. Poiché $f(1) = \operatorname{sen}\left(e^{\cos 1}\operatorname{arctg} 0\right) = 0$

$$f'(x) = \cos\left(e^{x\cos x}(\cos x - x\operatorname{sen} x)\operatorname{arctg}\lg x + \frac{e^{x\cos x}}{1+\lg^2 x}\cdot\frac{1}{x}\right)$$

$$f'(1) = \cos\left(e^{\cos 1}\cdot(\cos 1-\operatorname{sen} 1)\cdot 0 + \frac{e^{\cos 1}}{1+0}\cdot 1\right) = \cos\left(e^{\cos 1}\right)$$

la retta tangente ha equazione

$$y - f(1) = f'(1)(x-1) \quad \text{cioè} \quad y = \cos\left(e^{\cos 1}\right)(x-1)$$

6. Poiché $$-\frac{2}{1+i} - 10\,\frac{2+i}{2-i} = \frac{-4+2i-20-10i}{(1+i)(2-i)} = \frac{-8i-24}{2-i+2i-i^2} =$$

$$= \frac{-8\,\cancel{(3+i)}}{\cancel{3+i}} = 8\left(\cos\pi + i\,\operatorname{sen}\pi\right)$$

segue che

$$\bar{\omega}_K = \sqrt[5]{8\left(\cos\pi + i\,\operatorname{sen}\pi\right)} = \left\{\sqrt[5]{8}\left(\cos\frac{\pi+2K\pi}{5} + i\,\operatorname{sen}\frac{\pi+2K\pi}{5}\right) \quad K=0,1,\dots 4\right\}$$

e quindi

$$\omega_K = \sqrt[5]{8}\left(\cos\frac{\pi+2K\pi}{5} - i\,\operatorname{sen}\frac{\pi+2K\pi}{5}\right), \quad K = 0,1,2,3,4, \quad \text{cioè}$$

$$\omega_0 = \sqrt[5]{8}\left(\cos\frac{\pi}{5} - i\,\operatorname{sen}\frac{\pi}{5}\right)$$

$$\omega_1 = \sqrt[5]{8}\left(\cos\frac{3}{5}\pi - i\,\operatorname{sen}\frac{3}{5}\pi\right)$$

$$\omega_2 = \sqrt[5]{8}\left(\cos\pi - i\,\operatorname{sen}\pi\right) = -\sqrt[5]{8}$$

$$\omega_3 = \sqrt[5]{8}\left(\cos\frac{7}{5}\pi - i\,\operatorname{sen}\frac{7}{5}\pi\right)$$

$$\omega_4 = \sqrt[5]{8}\left(\cos\frac{9}{5}\pi - i\,\operatorname{sen}\frac{9}{5}\pi\right).$$