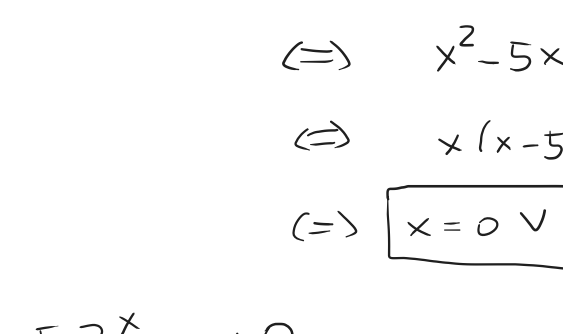


Exercises: Sheet 4

1.a) $A = (-\infty, 0)$, $A^c = [0, \infty)$ ($A^c = \mathbb{R} \setminus A$).

1.b) $B = (-\infty, -1) \cup (2, +\infty)$, $C = (-2, 0) \cup \{4\} \Rightarrow B \cap C = (-2, -1) \cup \{4\}$



2.a) $\log(1 - |x+3|) = 1 \Leftrightarrow e^{\log(1 - |x+3|)} = e^1 \Leftrightarrow 1 - |x+3| = e \Leftrightarrow 1 - e = |x+3|$

$1 - e < 0$, $|x+3| \geq 0 \quad \forall x \Rightarrow \nexists$ solutions.

alternative way:

$$\begin{cases} 1 - (x+3) = e \\ x+3 > 0 \end{cases} \vee \begin{cases} 1 + (x+3) = e \\ x+3 < 0 \end{cases} \Leftrightarrow \begin{cases} 1 - e = x+3 \\ x+3 > 0 \end{cases} \vee \begin{cases} x+4 = e \\ x < -3 \end{cases} \Leftrightarrow \begin{cases} x = -2 - e \\ x > -3 \end{cases} \vee \begin{cases} x = e - 4 \\ x < -3 \end{cases}$$

impossible impossible

2.b) $|x^2 - 2x + 1| = 3x \Leftrightarrow x^2 - 2x = 3x \vee x^2 - 2x = -3x$

$\Leftrightarrow x^2 - 5x = 0 \vee x^2 + x = 0$

$\Leftrightarrow x(x-5) = 0 \vee x(x+1) = 0 \Leftrightarrow (x=0 \vee x=5) \vee (x=0 \vee x=-1)$

$\Leftrightarrow \boxed{x=0 \vee x=1 \vee x=5}$

3.c) $9^x - 5 \cdot 3^x + 4 \leq 0$

$3^x = y$

$y_1 = \frac{5+3}{2} = 4 \quad y_2 = \frac{5-3}{2} = 1$

$y^2 - 5y + 4 \leq 0, y > 0 \Rightarrow y \in [1, 4]$

$a=1 \quad b=5 \quad c=4$

$D = 25 - 16 = 9$

$3^x \in [1, 4] \Leftrightarrow 1 \leq 3^x \leq 4 \Leftrightarrow 3^x \geq 1 \wedge 3^x \leq 4 \Leftrightarrow \log_3(3^x) \geq \log_3 1 \wedge \log_3 3^x \leq \log_3 4$

$\Leftrightarrow x \geq 0 \wedge x \leq \log_3 4 \Leftrightarrow x \in [0, \log_3 4]$.

Another example:

$9^x - 3 \cdot 3^x - 4 \geq 0; y = 3^x, \begin{cases} y^2 - 3y - 4 \geq 0 \\ y > 0 \end{cases} \quad y_{\pm} = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} = 4/-1$

$\Leftrightarrow \begin{cases} y \leq -1 \\ y \geq 4 \end{cases} \vee y \geq 4 \quad \begin{array}{c} -1 \quad 4 \\ \hline 0 \end{array} \quad \Leftrightarrow y \geq 4 \Leftrightarrow 3^x \geq 4 \Leftrightarrow \boxed{x \geq \log_3 4}$

Another example: $x^4 - 3x^2 - 4 \geq 0 \quad y = x^2 \dots$ [exercise]

$\begin{cases} y^2 - 3y - 4 \geq 0 \\ y > 0 \end{cases} \Leftrightarrow y \geq 4 \Leftrightarrow x^2 \geq 4 \Leftrightarrow \boxed{x \leq -2 \vee x \geq 2} \Leftrightarrow |x| \geq 2$



2.c) $2\cos^2 x - 3\cos x + 1 > 0, y = \cos x$

$\begin{cases} 2y^2 - 3y + 1 > 0 \\ -1 \leq y \leq 1 \end{cases} \Leftrightarrow \begin{cases} y < \frac{1}{2} \vee y > 1 \\ -1 \leq y \leq 1 \end{cases} \Leftrightarrow -1 \leq y < \frac{1}{2} \Leftrightarrow -1 \leq \cos x < \frac{1}{2}$

$2y^2 - 3y + 1 > 0 \Leftrightarrow y < \frac{1}{2} \vee y > 1$

$y_{\pm} = \frac{3 \pm \sqrt{9-4 \cdot 2}}{2 \cdot 2} = \frac{3 \pm \sqrt{1}}{4} = \frac{3 \pm 1}{4} = \frac{2}{4} = \frac{1}{2}$



$2y^2 - 3y + 1 = 2(y - \frac{1}{2})(y - 1) > 0$

$y = \frac{1}{2}$

$-1 \leq \cos x < \frac{1}{2}$



$x \in \bigcup \left(\frac{\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k \right)$

3) Consider the function $f(x) = \log\left(\frac{|x-1|}{x^2+1}\right)$

a) Determine the domain $D(f)$ of f .

$\frac{|x-1|}{x^2+1} > 0 \quad \forall x \in \mathbb{R} \setminus \{1\} \Rightarrow D(f) = \mathbb{R} \setminus \{1\}$.

b) Determine the zeros of f . This means: solve equation $f(x) = 0$.

$\log_3\left(\frac{|x-1|}{x^2+1}\right) = 0 \Leftrightarrow 3^{\log_3\left(\frac{|x-1|}{x^2+1}\right)} = 3^0 \Leftrightarrow \frac{|x-1|}{x^2+1} = 1, x \in D(f)$

$\Leftrightarrow \begin{cases} \frac{|x-1|}{x^2+1} = 1 \\ x \neq 1 \end{cases} \Leftrightarrow \begin{cases} |x-1| = x^2+1 \\ x \neq 1 \end{cases} \Leftrightarrow \begin{cases} x-1 = x^2+1 \vee -(x-1) = x^2+1 \\ x \neq 1 \end{cases}$

$\Leftrightarrow \begin{cases} x^2 - x + 2 = 0 \vee x^2 + x = 0 \\ x \neq 1 \end{cases} \Leftrightarrow \begin{cases} x = -1 \vee x = 0 \\ x \neq 1 \end{cases} \Leftrightarrow \boxed{x = -1 \vee x = 0}$

$x^2 - x + 2 = 0 \quad ; \quad x^2 + x = 0 \Leftrightarrow x = -1 \vee x = 0$

$\Delta = 1 - 4 \cdot 2 < 0 \quad ; \quad x^2 + x = x(x+1)$

Answer: the zeros of f are $x = -1$ and $x = 0$.

c) Is $f: D(f) \rightarrow \mathbb{R}$ injective? No, because $f(0) = f(-1) = 0$.

Recall: $f: X \rightarrow Y$ is injective if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$; f is surjective if $\forall y \in Y, \exists x \in X, y = f(x)$

f is bijective if f is both injective and surjective.

d) Determine $f^{-1}(-\infty, 1)$.

If I ask you to compute $f^{-1}(A)$, with $A \subseteq \mathbb{R}$ (A is a set), don't compute the inverse of f , but just the inverse image of the set A through f .

$f^{-1}(-\infty, 1) = \{x \in D(f) : f(x) \in (-\infty, 1)\} = \{x \in D(f) : f(x) < 1\}$

$\left[\begin{array}{l} f^{-1}(A) = \{x \in D(f) : f(x) \in A\} \\ A \subseteq \mathbb{R} \end{array} \right] \quad \text{So we need to solve inequality } f(x) < 1$

$\log_3\left(\frac{|x-1|}{x^2+1}\right) < 1 \Leftrightarrow 3^{\log_3\left(\frac{|x-1|}{x^2+1}\right)} < 3^1$

$\Leftrightarrow \frac{|x-1|}{x^2+1} < 3 \Leftrightarrow \frac{|x-1|}{x^2+1} - 3 < 0$

$\Leftrightarrow \frac{|x-1| - 3(x^2+1)}{x^2+1} < 0 \Leftrightarrow |x-1| - 3(x^2+1) < 0$

$\Leftrightarrow |x-1| < 3(x^2+1) \Leftrightarrow x-1 < 3(x^2+1) \vee -(x-1) < 3(x^2+1)$

$\Leftrightarrow x-1 < 3x^2+3 \vee 1-x < 3x^2+3 \Leftrightarrow 3x^2 - x + 4 > 0 \vee 3x^2 + x + 2 > 0$

$3x^2 - x + 4 > 0 \quad \forall x \in \mathbb{R} \quad (\Delta < 0)$

$3x^2 + x + 2 > 0 \quad \forall x \in \mathbb{R} \quad (\Delta < 0)$

$\log_3\left(\frac{|x-1|}{x^2+1}\right) < 1 \quad \forall x \in D(f) = \mathbb{R} \setminus \{1\} \Rightarrow f^{-1}(1, +\infty) = D(f) = \mathbb{R} \setminus \{1\}$.

4.a) Let $f(x) = \log(\sqrt{x+1})$. Determine the domain $D(f)$ of f .

$\sqrt{x+1} > 0 \Leftrightarrow x+1 > 0 \Leftrightarrow \boxed{x > -1} \quad ; \quad D(f) = (-1, \infty)$

b) Say if $f: D(f) \rightarrow f(D(f))$ is invertible or not and, in case it is invertible, compute the inverse.

Take $y \in \mathbb{R}$ and consider equation $y = \log(\sqrt{x+1}) \quad [y = f(x)]$

Let's see for which y such an equation is solvable and how many solutions we get. If $\forall y \in Y \subseteq \mathbb{R}$ we get a unique solution $\Rightarrow f: D(f) \rightarrow Y$ is invertible and $f^{-1}(y)$ is the solution to eq $f(x) = y$.

Let $y \in \mathbb{R}$, we want to solve $\log(\sqrt{x+1}) = y \Leftrightarrow e^{\log(\sqrt{x+1})} = e^y$

$\Leftrightarrow \sqrt{x+1} = e^y \Leftrightarrow \begin{cases} 1+x = e^{2y} \\ 1+x > 0 \end{cases} \Leftrightarrow 1+x = e^{2y} \Leftrightarrow \boxed{x = e^{2y} - 1}$

$\forall y \in \mathbb{R}, \exists! y \in (-1, \infty) : f(x) = y$; in particular $x = e^{2y} - 1$

This means that $f: D(f) \rightarrow f(D(f)) = \mathbb{R}$ is invertible, with inverse

$f^{-1}: \mathbb{R} \rightarrow D(f) = (-1, \infty)$

$f^{-1}(y) = e^{2y} - 1, \forall y \in \mathbb{R}$.

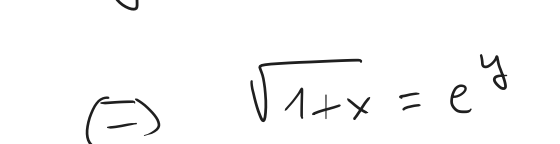
Limits:

Limits from the right and from the left:

Given a point $x_0 \in \mathbb{R}$, we say that a set $J \subseteq \mathbb{R}$ is a right neighbourhood of x_0 if $\exists$ a neighbourhood $I \subseteq \mathbb{R}$ of x_0 : $J = I \cap (x_0, \infty)$. Similarly, we say that a set $J \subseteq \mathbb{R}$ is a left neighbourhood of x_0 if $\exists$ a neighbourhood I of x_0 s.t.

$J = I \cap (-\infty, x_0)$.

In other words, a right neighbourhood of x_0 is a set of the form $(x_0, x_0 + \delta)$, for some $\delta > 0$. Similarly a left neighbourhood of x_0 is a set of the form $(x_0 - \delta, x_0)$, for some $\delta > 0$.



Definition (Limit from the right or from the left).

Let $f: X \subseteq \mathbb{R} \rightarrow \mathbb{R}$ let $x_0 \in \mathcal{A}(X) \cap \mathbb{R}$ and $y_0 \in \mathbb{R}$. We say that

$\lim_{x \rightarrow x_0^+} f(x) = y_0$ if $\forall$ neighbourhood $J \subseteq \mathbb{R}$ of y_0 , $\exists$ a right (or left) neighbourhood $I \subseteq \mathbb{R}$ of x_0 ; $\forall x \in (X \cap I) \setminus \{x_0\} \neq \emptyset$, we have $f(x) \in J$.

Example: $f: (0, \infty) \rightarrow \mathbb{R}$

$x \mapsto \log x$



$\lim_{x \rightarrow 0^+} \log x = -\infty$

we cannot compute $\lim_{x \rightarrow 0^+} \log x$.

Equivalent definition:

$y_0 \in \mathbb{R}, \lim_{x \rightarrow x_0^+} f(x) = y_0$.

$\forall \varepsilon > 0, \exists \delta > 0: \forall x \in X, x_0 < x < x_0 + \delta$, we have $|f(x) - y_0| < \varepsilon$.

Example: $f: (0, \infty) \rightarrow \mathbb{R}$

$x \mapsto 2x$



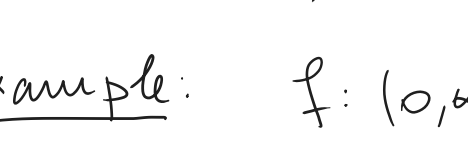
$\lim_{x \rightarrow 0^+} f(x) = 0$

$y_0 = +\infty, \lim_{x \rightarrow x_0^+} f(x) = +\infty$.

$\forall K > 0, \exists \delta > 0: \forall x \in X, x_0 < x < x_0 + \delta$, we have $f(x) > K$

take $f(x) = \frac{1}{x}$. $D(f) = \mathbb{R} \setminus \{0\}$. $0 \in \mathcal{A}(D(f))$

the graph is like this



$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty, \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$

Remark: the limit from the left and the limit from the right may both exist, but they may be different.

Proposition: If $f: X \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a function, $x_0 \in \mathcal{A}(X)$, $\lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x) =: \alpha \in \overline{\mathbb{R}}$

$\Rightarrow \lim_{x \rightarrow x_0} f(x) = \alpha$.

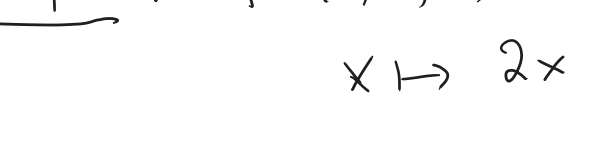
If $\lim_{x \rightarrow x_0^+} f(x) \neq \lim_{x \rightarrow x_0^-} f(x) \Rightarrow \nexists \lim_{x \rightarrow x_0} f(x)$.

Example: $\nexists \lim_{x \rightarrow 0} \frac{1}{x}$.

Remark: even if $x_0 \in \mathcal{A}(X)$, $\lim_{x \rightarrow x_0} f(x)$ does not always exist.

Example: let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a periodic function [f is periodic of period $T > 0$ if $f(x+T) = f(x), \forall x \in \mathbb{R}$]. If $\exists \lim_{x \rightarrow +\infty} f(x) \Rightarrow f$ is constant

(or $\exists \lim_{x \rightarrow -\infty} f(x)$)

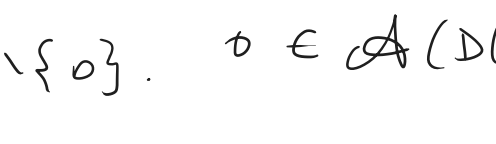


$f(x) = \sin x$

Definition (Vertical asymptotes) let $f: X \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a function. A vertical straight line of the form $\{x = x_0\}$, with $x_0 \in \mathbb{R}$, is said to be a vertical asymptote of f if either $\exists \lim_{x \rightarrow x_0^+} f(x) \in \overline{\mathbb{R}} \setminus \mathbb{R} = \{\pm\infty\}$

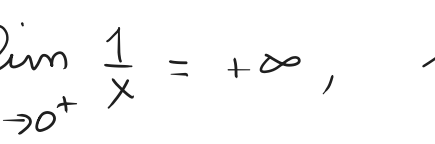
or $\exists \lim_{x \rightarrow x_0^-} f(x) \in \overline{\mathbb{R}} \setminus \mathbb{R} = \{\pm\infty\}$.

Example $f(x) = \log x$



$\{x=0\}$ is a vertical asymptote of $\log x$.

$f(x) = \frac{1}{x}$



$\{x=0\}$ is a vertical asymptote of $\frac{1}{x}$.

$f(x) = \frac{1}{x^2}; D(f) = \mathbb{R} \setminus \{0\}$



$\lim_{x \rightarrow 0^+} \frac{1}{x^2} = +\infty; \lim_{x \rightarrow 0^-} \frac{1}{x^2} = +\infty$

In particular, $\lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty$

$\{x=0\}$ is a vertical asymptote of $\frac{1}{x^2}$

$\{y=0\}$ is a horizontal asymptote of $\frac{1}{x^2}$.

2 Questions: Can the graph of f intersect a horizontal asymptote?

$f(x) = \frac{\sin x}{x}; \lim_{x \rightarrow +\infty} \frac{\sin x}{x} = 0$



$|\frac{\sin x}{x}| \leq \frac{1}{|x|}$

Can the graph of f intersect a vertical asymptote?

generally not

If we set $f(x) = \begin{cases} 1/x & x \neq 0 \\ 1 & x = 0 \end{cases}$

then yes.

