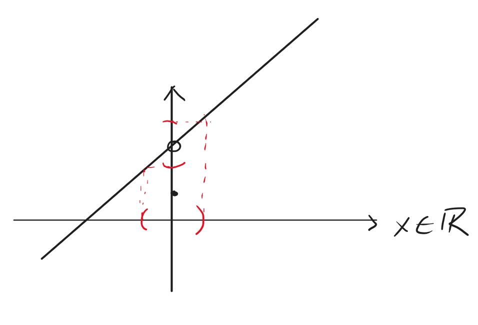
# Limits:

**Def:** Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$. Let $x_0 \in \mathcal{A}(X)$ and $y_0 \in \overline{\mathbb{R}}$. We say that $\lim_{x \to x_0} f(x) = y_0$ if $\forall$ neighbourhood $J \subseteq \mathbb{R}$ of $y_0$, $\exists$ a neighbourhood $I \subseteq \mathbb{R}$ of $x_0$ s.t. $\forall x \in (X \cap I) \setminus \{x_0\}$, we have $f(x) \in J$.

**Example:** assume that $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$.

Equivalent formulation of the definition of limit:
we say that $y_0 = \lim_{x \to x_0} f(x)$ if $\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in X \setminus \{x_0\}$, with $|x - x_0| < \delta$, we have $|f(x) - y_0| < \varepsilon$.

$$f(x) = \begin{cases} 2x+3 & x \neq 0 \\ 1 & x = 0 \end{cases}$$




**claim** $\lim_{x \to 0} f(x) = 3$ ; fix $\varepsilon > 0$.

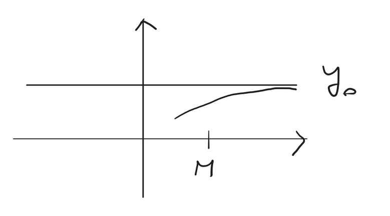
$|f(x) - 3| < \varepsilon$, $x \neq 0 \iff |2x + \cancel{3} - \cancel{3}| < \varepsilon$, $x \neq 0 \iff |2x| < \varepsilon$, $x \neq 0 \iff -\varepsilon < 2x < \varepsilon$, $x \neq 0$

$\iff -\frac{\varepsilon}{2} < x < \frac{\varepsilon}{2}$, $x \neq 0$; $\delta := \frac{\varepsilon}{2} > 0$

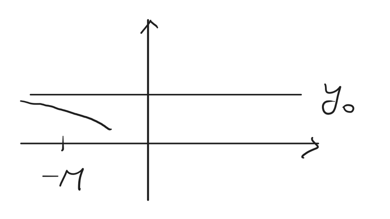
we have proved that $\forall x \in (-\delta, \delta) \setminus \{0\}$, $|f(x) - 3| < \varepsilon$.

**Other case:** (horizontal asymptots) $y_0 \in \mathbb{R}$, $x_0 \in \{\pm\infty\}$. $[f : X \subseteq \mathbb{R} \to \mathbb{R}]$

- we say that $\lim_{x \to +\infty} f(x) = y_0$ if
  $\forall \varepsilon > 0\ \exists M > 0 : \forall x \in X, x > M$, we have $|f(x) - y_0| < \varepsilon$.

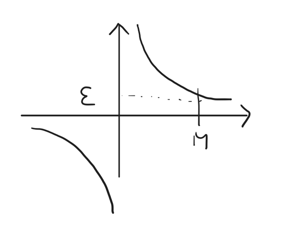



- we say that $\lim_{x \to -\infty} f(x) = y_0$ if
  $\forall \varepsilon > 0\ \exists M > 0 : \forall x \in X, x < -M$, we have $|f(x) - y_0| < \varepsilon$.




- For $y_0 \in \mathbb{R}$, we say that the straight line $\{y = y_0\}$ is a horizontal asymptot of $f$ if
  either $\lim_{x \to +\infty} f(x) = y_0$ or $\lim_{x \to -\infty} f(x) = y_0$.

**Examples:** $f(x) = \frac{1}{x}$; $D(f) = \mathbb{R} \setminus \{0\}$; $\mathcal{A}(D(f)) = \overline{\mathbb{R}}$.
$\mathcal{A}(D(f)) \setminus D(f) = \{\pm\infty\} \cup \{0\}$.




$\lim_{x \to +\infty} f(x) = 0$.

fix $\varepsilon > 0$. we need to find $M > 0$: $\forall x > M$, we have $|f(x)| < \varepsilon$.
given $\varepsilon > 0$, we consider inequality $\left|\frac{1}{x}\right| < \varepsilon$, $x > 0 \iff \frac{1}{x} < \varepsilon$, $x > 0 \iff x > \frac{1}{\varepsilon} =: M$
take $M := \frac{1}{\varepsilon} > 0 \implies$ ok
$\forall x > M = \frac{1}{\varepsilon}$, we have $|f(x)| = \left|\frac{1}{x}\right| = \frac{1}{x} < \varepsilon$.

**Intuition:** if $x$ is very large, then $\frac{1}{x}$ is very small. $\left[\text{"}\frac{1}{+\infty} = 0\text{"}\right]$

$\lim_{x \to -\infty} \frac{1}{x} = 0$

$\left|\frac{1}{x}\right| < \varepsilon$, $x < 0 \iff -\frac{1}{x} < \varepsilon \underset{x<0}{\iff} \frac{1}{x} > -\varepsilon \iff x < -\frac{1}{\varepsilon}$, $M = \frac{1}{\varepsilon} > 0$ ok

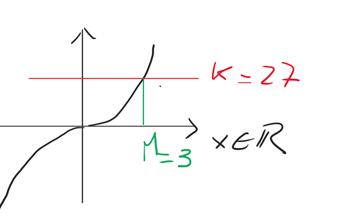
- Infinite limits as $x \to \pm\infty$:

we say that $\lim_{x \to +\infty} f(x) = +\infty$ if $\forall k > 0, \exists M > 0 : \forall x \in X, x > M$, we have $f(x) > k$.
$\lim_{x \to +\infty} f(x) = -\infty$ if $\forall k > 0, \exists M > 0 : \forall x \in X, x > M$, we have $f(x) < -k$.

**Examples:** $f(x) = x^3$. $\lim_{x \to +\infty} f(x) = +\infty$.

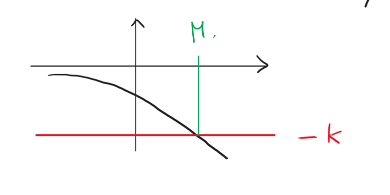
fix $k > 0$. Consider inequality $x^3 > k \iff x > \sqrt[3]{k} =: M$.

$x^3 > 27 \iff x > \sqrt[3]{27} = 3$



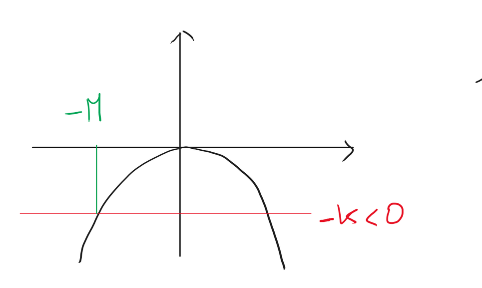

"if $x$ is very large, then $x^3$ is also very large"

$f(x) = -e^x$, $\lim_{x \to +\infty} f(x) = -\infty$.




- we say that $\lim_{x \to -\infty} f(x) = +\infty$ if $\forall k > 0, \exists M > 0 : \forall x \in X, x < -M$, we have $f(x) > k$.
  $\lim_{x \to -\infty} f(x) = -\infty$ if $\forall k > 0, \exists M > 0 : \forall x \in X, x < -M$, we have $f(x) < -k$.

$f(x) = -x^2$




$\lim_{x \to -\infty} (-x^2) = -\infty$

$M = \sqrt{k}$

$\forall x < -\sqrt{k}$, $-x^2 < k$

## Exercises:

Sheet 3, exercise 2,c.

$$\frac{\sqrt{x^2+3} - x - 1}{\sqrt{x}\,(x+2)} \geq 0 \quad \Rightarrow x \to -2, 0, 1$$

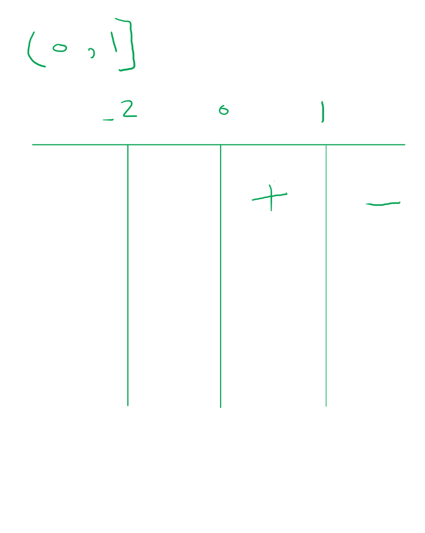
- $N(x) \geq 0$.

$\sqrt{x} > 0$
$x > 0$ (1)

$\sqrt{x^2+3} - x - 1 \geq 0$

$(\sqrt{x^2+3} \geq x+1)^2 \to x^2 + 3 \geq x^2 + 2x + 1$

$2 \geq 2x \to 1 \geq x$




- Existence conditions: $\underset{\forall x \in \mathbb{R}}{x^2 + 3 \geq 0}$, $x > 0$, $x \neq -2$ $(\iff x+2 \neq 0)$

our fraction is defined for $\boxed{x > 0}$

- $N(x) \geq 0 \iff \sqrt{x^2+3} \geq x+1 \iff x^2 + 3 \geq x^2 + 1 + 2x \iff 2x \leq 2 \iff x \leq 1$
- $D(x) > 0 \iff \sqrt{x}(x+2) > 0 \iff x > 0 \wedge x + 2 > 0 \iff x > 0 \wedge x > -2 \iff x > 0$

| 0 | | 1 |
|---|---|---|
| | + | − |
| | + | + |
| | + | − |

$$\frac{\sqrt{x^2+3} - x - 1}{\sqrt{x}\,(x+2)} \geq 0 \iff 0 < x \leq 1.$$

## Sheet 2, exercise 4,a.

$$\begin{cases} \sqrt{x^2+2x} \leq 2 \\ e^{2x} - 4 > 0 \end{cases} \iff \begin{cases} x^2 + 2x \geq 0 \ \wedge \ x^2 + 2x \leq 4 & \text{I} \\ e^{2x} > 4 & \text{II} \end{cases}$$

I $\begin{cases} x^2 + 2x \geq 0 \\ x^2 + 2x - 4 \leq 0 \end{cases} \iff \begin{cases} x(x+2) \geq 0 \\ x^2 + 2x - 4 \leq 0 \end{cases} \iff \begin{cases} x \leq -2 \vee x \geq 0 \\ -1-\sqrt{5} \leq x \leq -1+\sqrt{5} \end{cases}$

- $x(x+2) = 0 \iff x = 0 \vee x + 2 = 0 \iff x = 0 \vee x = -2$
  $x(x+2) \geq 0 \iff x \leq -2 \vee x \geq 0$.

| | −2 | | 0 | |
|---|---|---|---|---|
| $x$ | − | − | | + |
| $x+2$ | − | + | | + |
| | + | − | | + |

- $x^2 + 2x - 4 \leq 0 \iff -1-\sqrt{5} \leq x \leq -1+\sqrt{5}$

$$x_\pm = \frac{-2 \pm \sqrt{4+16}}{2} = \frac{-2 \pm \sqrt{20}}{2} = \frac{-2 \pm \sqrt{4 \cdot 5}}{2} = -1 \pm \sqrt{5}$$

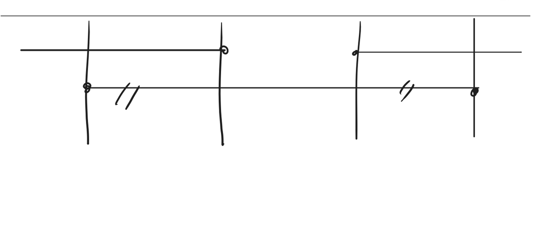
$x^2 + 2x - 4 = (x - x_-)(x - x_+) \leq 0$

| | $-1-\sqrt{5}$ | | $-1+\sqrt{5}$ | |
|---|---|---|---|---|
| $x-(-1-\sqrt{5})$ | − | + | | + |
| $x-(-1+\sqrt{5})$ | − | − | | + |
| | + | − | | + |

$\sqrt{5} > \sqrt{4} = 2$
$\Rightarrow -1 + \sqrt{5} > -1 + 2 = 1 > 0$

$x - x_- > 0 \iff x > x_-$

I $\iff \begin{cases} x \leq -2 \vee x \geq 0 \\ -1-\sqrt{5} \leq x \leq -1+\sqrt{5} \end{cases}$



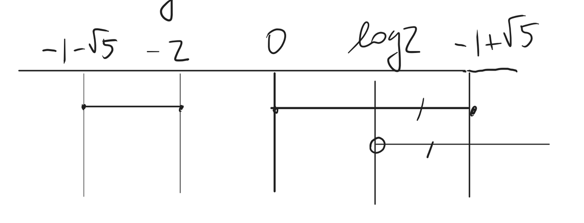

$-1-\sqrt{5} < -1-\sqrt{4} = -3 < -2$

$\iff -1-\sqrt{5} \leq x \leq -2 \ \vee \ 0 \leq x \leq -1+\sqrt{5}$.

II $e^{2x} > 4 \iff \log e^{2x} > \log 4 \iff 2x > \log 4 = \log(2^2) = 2\log 2 \iff x > \log 2$

$$\begin{cases} -1-\sqrt{5} \leq x \leq -2 \ \vee \ 0 \leq x \leq -1+\sqrt{5} \\ x > \log 2 \end{cases}$$

$-1+\sqrt{5} > -1 + 2 = 1 = \log e > \log 2$




final answer $\boxed{\log 2 < x \leq -1+\sqrt{5}}$