Exercises

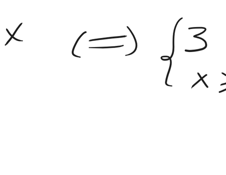
Sheet 3, ex 2.b

$$\frac{\sqrt{x^2-2x+4}-|x+1|}{\sqrt[3]{x+6}-\sqrt[3]{x}} \leq 0$$

$N(x)=\sqrt{x^2-2x+4}-|x+1|\leq 0 \iff \sqrt{x^2-2x+4}\leq |x+1| \iff \begin{cases} x+1\geq\sqrt{x^2-2x+4} \ \vee\ x+1\leq-\sqrt{x^2-2x+4} & \text{I} \\ x^2-2x+4\geq 0 & \text{II}\end{cases}$

II $x^2-2x+4\geq 0 \quad \forall x\in\mathbb{R}$

$x_\pm=\frac{2\pm\sqrt{4-16}}{2}$ impossible $(\Delta<0)$



$ax^2+bx+c=0 \quad (a\neq 0) \iff x=x_\pm=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

if $\Delta=b^2-4ac\geq 0$

if $\Delta<0 \Rightarrow \nexists$ solution.

I $0\leq\sqrt{x^2-2x+4}\leq x+1 \ \vee\ \sqrt{x^2-2x+4}\leq -x-1$

$\sqrt{x^2-2x+4}\leq x+1 \iff \begin{cases} x^2-2x+4\leq x^2+1+2x \\ x+1\geq 0\end{cases} \iff \begin{cases}3\leq 4x \\ x\geq -1\end{cases} \iff \begin{cases} x\geq 3/4 \\ x\geq -1\end{cases} \iff x\geq\frac{3}{4}$

$(\sqrt{a}\leq b \Rightarrow b\geq 0)$

$0\leq\sqrt{x^2-2x+4}\leq -x-1 \iff \begin{cases} x^2-2x+4\leq x^2+1+2x \\ -x-1\geq 0\end{cases} \iff \begin{cases} x\geq 3/4 \\ x\leq -1\end{cases}$ impossible.

$(-x-1)^2=(-(x+1))^2=(-1)^2(x+1)^2=(x+1)^2$

$\begin{cases}\text{I} \\ \text{II}\end{cases} \iff \begin{cases} x\geq 3/4 \\ x\in\mathbb{R}\end{cases} \iff x\geq 3/4$. Finally $N(x)\leq 0 \ \forall x\geq 3/4$.

- $D(x)=\sqrt[3]{x+6}-\sqrt[3]{x}$

Remark: $\sqrt[3]{\ }$ is increasing on $\mathbb{R} \Rightarrow \sqrt[3]{y}>\sqrt[3]{x}$ for $y>x$.

$x+6>x, \forall x\in\mathbb{R} \Rightarrow \sqrt[3]{x+6}>\sqrt[3]{x} \iff D(x)>0$

- Another way to solve the ineq $D(x)>0$.

$D(x)>0 \iff (\sqrt[3]{x+6})^3>(\sqrt[3]{x})^3 \iff x+6>x$. $\forall x\in\mathbb{R}$. So $D(x)>0 \ \forall x$

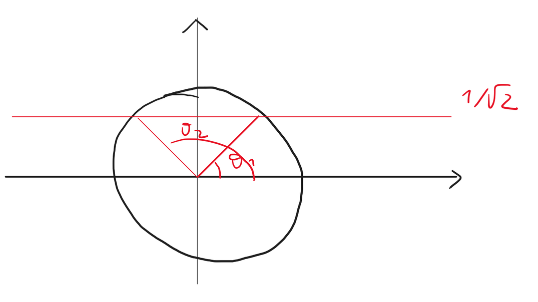
- The final solution of $\frac{N(x)}{D(x)}\leq 0$ is $\boxed{x\geq 3/4}$.

- Sheet 3, exercise 1,d.

$(3-\tan^2 x)(\sqrt{2}\sin x-1)=0 \iff 3-\tan^2 x=0 \ \vee\ \sqrt{2}\sin x-1=0 \iff \tan^2 x=3 \ \vee\ \sin x=\frac{1}{\sqrt{2}}$

$\iff \tan x=\pm\sqrt{3} \ \vee\ \sin x=\frac{1}{\sqrt{2}}$

- $\sin x=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2} \iff x=\frac{\pi}{4}+2\pi k \ \vee\ x=\frac{3\pi}{4}+2\pi k, \ \forall k\in\mathbb{Z}$



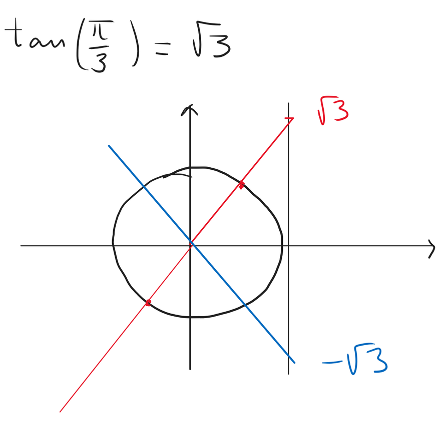
- $\tan x=\pm\sqrt{3}$

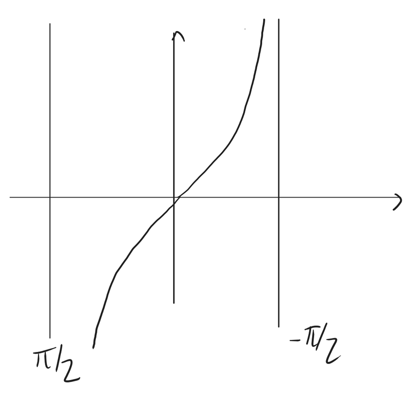
$\tan x=\frac{\sin x}{\cos x}$

Some known values of the tangent:

$\tan 0=0, \ \tan\frac{\pi}{6}=\frac{\sin(\pi/6)}{\cos(\pi/6)}=\frac{1/2}{\sqrt{3}/2}=\frac{1}{2}\cdot\frac{2}{\sqrt{3}}=\frac{1}{\sqrt{3}}; \ \tan\frac{\pi}{4}=\frac{\sin\pi/4}{\cos\pi/4}=1$

$\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$



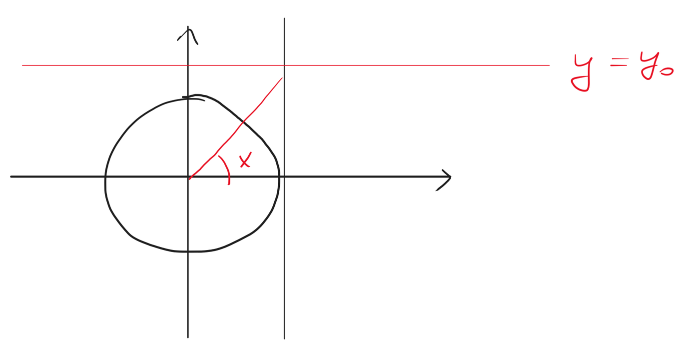


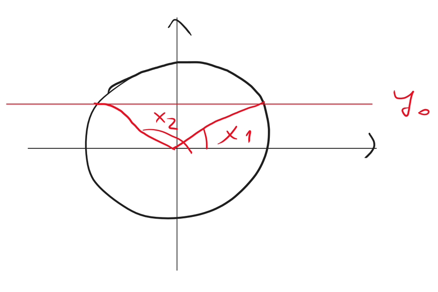
$\tan x=\sqrt{3} \iff x=\frac{\pi}{3}+k\pi$

$\tan x=-\sqrt{3} \iff x=-\frac{\pi}{3}+k\pi$

The solutions to the exercise are $x=\frac{\pi}{4}+2k\pi \ \vee\ x=\frac{3\pi}{4}+2k\pi \ \vee\ x=\frac{\pi}{3}+k\pi \ \vee\ x=-\frac{\pi}{3}+k\pi$.

- For example: if we have to solve $\tan x=y_0$ | $\sin x=y_0, \ y_0\in[-1,1]$



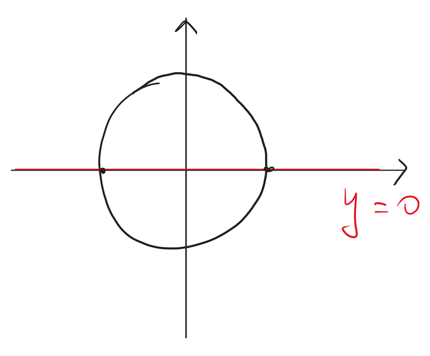


Sheet 3, ex 2,e:

$\sin 2x-2\sin x>0 \iff 2\sin x\cos x-2\sin x>0 \iff 2\sin x(\cos x-1)>0$

$[\sin 2x=2\sin x\cos x]$

- $\sin x>0 \iff x\in\bigcup_{k\in\mathbb{Z}}(0+2k\pi, \pi+2k\pi)=\bigcup_{k\in\mathbb{Z}}(2k\pi, \pi(1+2k))$



- $\cos x-1>0$ never

$2\sin x(\cos x-1)>0 \iff \sin x<0, \ \cos x-1<0 \iff x\in\bigcup_k(\pi(2k-1), 2\pi k)$

$\cos x-1<0 \iff \cos x<1 \iff x\neq 2\pi k$

- Sheet 3, ex 2,f

$4\sin^2(5x)\cos^2(5x)<1 \iff (2\sin(5x)\cos(5x))^2<1 \iff \sin(2\cdot 5x)^2<1$

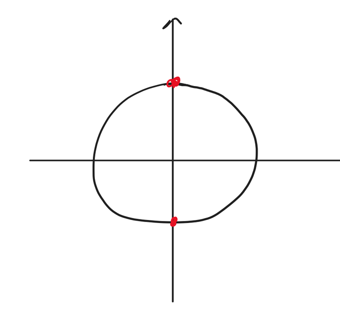
$\iff \sin(10x)^2<1 \iff -1<\sin(10x)<1 \iff x\in\mathbb{R}\setminus\left(\bigcup_{k\in\mathbb{Z}}\left\{\frac{\pi}{20}+\frac{\pi k}{5}\right\}\cup\bigcup_{k\in\mathbb{Z}}\left\{-\frac{\pi}{20}+\frac{\pi k}{5}\right\}\right)$

$\sin(10x)=1 \iff 10x=\frac{\pi}{2}+2\pi k \iff x=\frac{\pi}{20}+\frac{\pi k}{5}$

$\sin(10x)=-1 \iff 10x=-\frac{\pi}{2}+2\pi k \iff x=-\frac{\pi}{20}+\frac{\pi k}{5}$

the solution can be expressed by saying $x\neq\frac{\pi}{20}+\frac{\pi k}{5}$ and $x\neq-\frac{\pi}{20}+\frac{\pi k}{5}$

- equivalent way: $\sin(10x)\in(-1,1) \iff 10x=\frac{\pi}{2}+k\pi \iff x=\frac{\pi}{20}+\frac{k\pi}{10}$

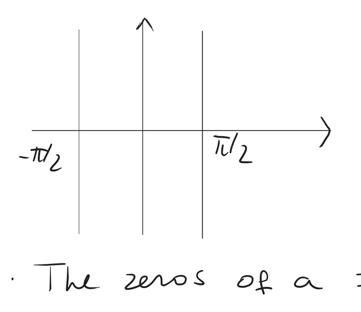


the solution to the exercise can be expressed by saying that $x\neq\frac{\pi}{20}+\frac{k\pi}{10}$

Sheet 3, ex 3: Determine the domain and the zeros of the function

$f(x)=\sin x(\tan x-1)$. Is $f: D(f)\to\mathbb{R}$ injective?

- $D(f)=\{x\in\mathbb{R}: x\neq\frac{\pi}{2}+k\pi, k\in\mathbb{Z}\}=\mathbb{R}\setminus\bigcup_{k\in\mathbb{Z}}\left\{\frac{\pi}{2}+k\pi\right\}$



- The zeros of a function $f: D(f)\to\mathbb{R}$ are the solutions to $f(x)=0$

$\sin x(\tan x-1)=0 \iff \sin x=0 \ \vee\ \tan x-1=0 \iff \boxed{x=k\pi \ \vee\ x=\frac{\pi}{4}+k\pi}$

- f is not injective. (Because equation $f(x)=0$ has more than 1 solutions)