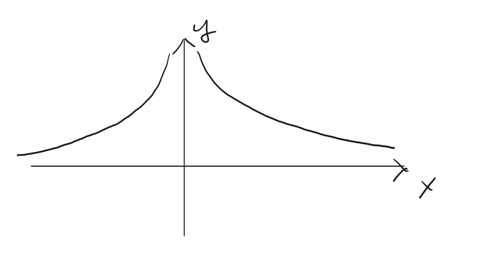
Limits:

$f(x)=\frac{1}{x^2}$, $\forall x\in\mathbb{R}\setminus\{0\}$



f is even.

"if x approaches 0, then $f(x)$ becomes larger and larger".
This idea can be expressed by saying that $\lim_{x\to 0}\frac{1}{x^2}=+\infty$.

Definition: $\overline{\mathbb{R}}:=\mathbb{R}\cup\{\pm\infty\}$.

Let $x\in\overline{\mathbb{R}}$. We say that an interval $I\subset\mathbb{R}$ is a neighbourhood of x if I is open and

(i) $x\in I$ if $x\in\mathbb{R}$

(ii) $I=(-\infty,a)$, with $a\in\mathbb{R}$ if $x=-\infty$

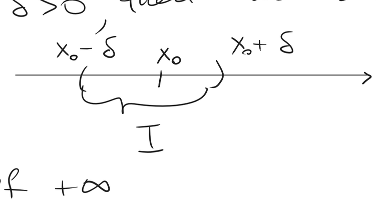
(iii) $I=(a,\infty)$, with $a\in\mathbb{R}$ if $x=+\infty$.

In other words, if $x\in\mathbb{R}$, a neighbourhood of x is any open interval containing x. If $x=+\infty$, a neighbourhood of x is any open half-line of the form $(a,+\infty)$. If $x=-\infty$, a neighbourhood of x is any open half-line of the form $(-\infty,a)$.

Definition (Accumulation points). Let $X\subseteq\mathbb{R}$ be a set. A point $x_0\in\overline{\mathbb{R}}$ is said to be an accumulation point of X if for any neighbourhood $I\subseteq\mathbb{R}$ of x_0 we have $(I\cap X)\setminus\{x_0\}\neq\phi$

. Given a set $X\subseteq\mathbb{R}$, the set of accumulation points of X will be denoted by $\mathcal{A}(X)$.

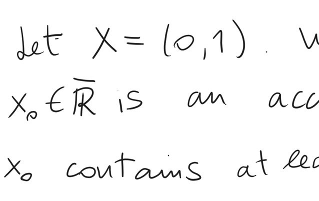
Examples: . Given a point $x_0\in\mathbb{R}$ and $\delta>0$, then the set $I=(x_0-\delta,x_0+\delta)$ is a neighbourhood of x_0



. $(0,+\infty)$ is a neighbourhood of $+\infty$

. $(-\infty,-1)$ is a neighbourhood of $-\infty$.

. Let $X=(0,1)$. What is $\mathcal{A}(X)$?
$x_0\in\overline{\mathbb{R}}$ is an accumulation point of X if any neighbourhood of x_0 contains at least 1 point of X which is different from x_0.



. $\forall x_0\in X$, $x_0\in\mathcal{A}(X)\Rightarrow X\subset\mathcal{A}(X)$.

. $x_0=0$, $x_0=1\Rightarrow x_0\in\mathcal{A}(X)$.

$x_0=0$, $I=(-\delta,\delta)$, $\delta>0$.

If $0<\delta<1\Rightarrow I\cap X=(0,\delta)\Rightarrow(I\cap X)\setminus\{0\}=(0,\delta)\neq\phi$

. $x_0\notin[0,1]\Rightarrow x_0\notin\mathcal{A}(X)$.

$\Rightarrow\mathcal{A}(X)=[0,1]$.

. $X=(0,1)\cup\{2\}\Rightarrow\mathcal{A}(X)=?$ $\quad\mathcal{A}(X)=[0,1]$; $x_0=0$, $\delta<\frac{1}{2}\Rightarrow(X\cap(-\delta,\delta))\setminus\{0\}=(0,\delta)\neq\phi$

$x_0=\frac{1}{2}$, $0<\delta<\frac{1}{2}$, $I=(x_0-\delta,x_0+\delta)\subseteq X$

$(I\cap X)\setminus\{x_0\}=I\setminus\{x_0\}=(x_0-\delta,x_0)\cup(x_0,x_0+\delta)\neq\phi$.

$x_0=2\in X$. $\delta=\frac{1}{2}$

$X\cap(x_0-\delta,x_0+\delta)=X\cap(\frac{3}{2},\frac{5}{2})=((0,1)\cup\{2\})\cap(\frac{3}{2},\frac{5}{2})=\{2\}=\{x_0\}$

$(X\cap(x_0-\delta,x_0+\delta))\setminus\{x_0\}=\phi$

. These examples show that the inclusion $X\subseteq\mathcal{A}(X)$ and $\mathcal{A}(X)\subseteq X$ are false.

Definition Let $X\subseteq\mathbb{R}$ be a set. We say that $x_0\in X$ is an isolated point if x_0 is not an accumulation point of X.

Remark: Given a set $X\subseteq\mathbb{R}$ and a point $x_0\in X$, either x_0 is an accumulation point of X or it is an isolated point.

Example: $X=(0,1)\cup\{2\}$. 2 is an isolated point of X.

Definition (Limit) Let $X\subseteq\mathbb{R}$ be a set. Let $f:X\to\mathbb{R}$ be a function. Let $y_0\in\overline{\mathbb{R}}$ and let $x_0\in\mathcal{A}(X)$. Then we say that $\lim_{x\to x_0}f(x)=y_0$ if

$\forall J\subseteq\mathbb{R}$ neighbourhood of y_0, $\exists$ a neighbourhood $I\subseteq\mathbb{R}$ of x_0:

$\forall x\in(X\cap I)\setminus\{x_0\}$, we have $f(x)\in J$.

Theorem (Uniqueness of the limit). Let $X\subseteq\mathbb{R}$ be a set, let $f:X\to\mathbb{R}$ be a function. Let $x_0\in\mathcal{A}(X)$. If $\exists\lim_{x\to x_0}f(x)$, then this limit is unique.

proof: Assume the limit is not unique. $y_0=\lim_{x\to x_0}f(x)$, $y_1=\lim_{x\to x_0}f(x)$, $y_0\neq y_1$.

$y_0\neq y_1\Rightarrow\exists$ neighbourhood J_0 and J_1 of y_0 and y_1: $J_0\cap J_1=\phi$.

$\Rightarrow\exists I_0$: $\forall x\in(I_0\cap X)\setminus\{x_0\}$, we have $f(x)\in J_0$.

$\exists I_1$: $\forall x\in(I_1\cap X)\setminus\{0\}$, we have $f(x)\in J_1$.

$I_2:=I_1\cap I_0\Rightarrow\forall x\in(I_2\cap X)\setminus\{0\}$, we have $f(x)\in J_0\cap J_1=\phi$. Contradiction $\square$

Exercises:

Sheet 1, exercise 5

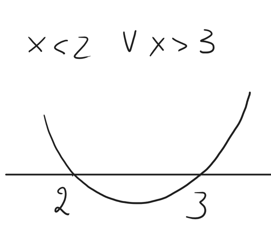
. Determine the domain of the function $f(x)=\log(x^2-5x+6)$ and determine $f^{-1}(\log 2,\infty)$.

Domain: $x^2-5x+6>0$ $\qquad x_\pm=\frac{5\pm\sqrt{25-24}}{2}=\frac{5\pm1}{2}=3/2$

$x^2-5x+6=a(x-x_-)(x-x_+)=(x-2)(x-3)>0$

$\Leftrightarrow x<2\ \vee\ x>3$

	2		3	
$x-2$	$-$	$+$		$+$
$x-3$	$-$	$-$		$+$
	$+$	$-$		$+$



$D(f)=(-\infty,2)\cup(3,+\infty)$ (Answer).

. What is $f^{-1}(\log 2,+\infty)$?

$f^{-1}(\log 2,+\infty)=\{x\in D(f): f(x)\in(\log 2,+\infty)\}=\{x\in D(f): f(x)>\log 2\}$

$\log(x^2-5x+6)>\log(2)$, $x\in D(f)$ $\quad[f(x)\in(\log 2,+\infty)\Leftrightarrow f(x)>\log 2]$

$e^{\log(x^2-5x+6)}>e^{\log 2}$ $\qquad(e>1)$

$\begin{cases}x^2-5x+6>2\\ x\in D(f)\end{cases}\Leftrightarrow\begin{cases}x^2-5x+4>0\\ x\in(-\infty,2)\cup(3,+\infty)\end{cases}\Leftrightarrow\begin{cases}x\in(-\infty,1)\cup(4,+\infty)\\ x\in(-\infty,2)\cup(3,+\infty)\end{cases}\Leftrightarrow x\in(-\infty,1)\cup(4,+\infty)$.

$x^2-5x+4>0\Leftrightarrow x>4\ \vee\ x<1$

$(x-4)(x-1)>0$

	1	4	
$x-4$	$-$	$-$	$+$
$x-1$	$-$	$+$	$+$
	$+$	$-$	$+$

$f^{-1}(\log 2,+\infty)=(-\infty,1)\cup(4,+\infty)$.

$f:X\subseteq\mathbb{R}\to\mathbb{R}$, $f^{-1}(Y)=\{x\in X: f(x)\in Y\}$

$Y\subseteq\mathbb{R}$

$Y=(\log 2,+\infty)$, $f^{-1}(Y)=(-\infty,1)\cup(4,+\infty)\subset X=(-\infty,2)\cup(3,+\infty)$.

$X=(-\infty,2)\cup(3,+\infty)$

$f^{-1}(\{\log 2\})=\{x\in D(f): f(x)\in Y\}=\{x\in D(f): f(x)=\log 2\}=\{1\}\cup\{4\}$

$Y=\{\log 2\}$

we need to solve $\log(x^2-5x+6)=\log 2\Leftrightarrow x^2-5x+6=2\Leftrightarrow x^2-5x+4=0$

$\Leftrightarrow x=1\vee x=4$.

. If $f(x)=2x$, $f^{-1}(0,+\infty)=\{f^{-1}(y): y>0\}=\{\frac{y}{2}: y>0\}=(0,+\infty)$

$\forall y\in\mathbb{R}$, we solve $f(x)=y\Leftrightarrow 2x=y\Leftrightarrow x=y/2\Rightarrow f^{-1}(y)=y/2$

What is $f^{-1}(0,\infty)$? Answer: we need to solve $f(x)>0$.

Sheet 3, ex 4: Determine the domains $D(f)$, $D(g)$ and the zeros of the functions

$f(x)=\frac{|2x-1|-2}{|2x+1|+2}$ and $g(x)=\sin(x^2+1)$. Determine $f^{-1}([0,+\infty))$.

Is $f:D(f)\to\mathbb{R}$ injective? Is $g:D(g)\to\mathbb{R}$ surjective

Answers: $D(f)=\mathbb{R}$, $D(g)=\mathbb{R}$.

$f^{-1}([0,+\infty))=\{x\in\mathbb{R}: f(x)\in[0,+\infty)\}=\{x\in\mathbb{R}: \frac{|2x-1|-2}{|2x+1|+2}\in[0,+\infty)\}=\{x\in\mathbb{R}: \frac{|2x-1|-2}{|2x+1|+2}\geq 0\}$

we need to solve $\frac{|2x-1|-2}{|2x+1|+2}\geq 0$. $\qquad a\in[0,+\infty)\Leftrightarrow a\geq 0$

this is equivalent to solve $|2x-1|-2\geq 0\Leftrightarrow|2x-1|\geq 2$

$\Leftrightarrow 2x-1\geq 2\ \vee\ 2x-1\leq -2$ $\quad\Leftrightarrow 2x\geq 3\ \vee\ 2x\leq -1\Leftrightarrow x\geq\frac{3}{2}\ \vee\ x\leq-\frac{1}{2}$

$-2x+1\geq 2$

$[|a(x)|\geq b(x)\Leftrightarrow a(x)\geq b(x)\vee a(x)\leq -b(x)$
$\Leftrightarrow a(x)\geq b(x)\vee -a(x)\geq b(x)]$

$\Rightarrow f^{-1}([0,\infty))=(-\infty,-\frac{1}{2}]\cup[\frac{3}{2},+\infty)$

. $|2x-1|-2=0\Leftrightarrow x=\frac{3}{2}\vee x=-\frac{1}{2}$; this means that $f(\frac{3}{2})=f(-\frac{1}{2})=0$.
$\Rightarrow f$ is not injective.

. g is not surjective, because $-1\leq\sin(x^2+1)\leq 1$, $\forall x\in\mathbb{R}$.
$g(x)=\sin(x^2+1)$, $g:\mathbb{R}\to\mathbb{R}$ is not surjective; $g:\mathbb{R}\to[-1,1]$ is surjective.

. $f(x)=\log\left(\frac{|2x-1|-2}{|2x+1|+2}\right)$. $\qquad D(f)=\{x\in\mathbb{R}: \frac{|2x-1|-2}{|2x+1|+2}>0\}=(-\infty,-\frac{1}{2})\cup(\frac{3}{2},+\infty)$