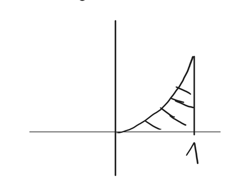
Integrals:

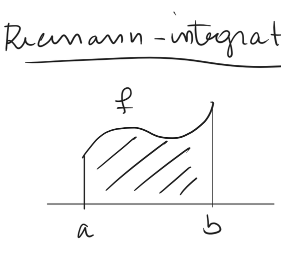


· Indefinite integration: given a function $f: I \to \mathbb{R}$, we want to find a primitive, that a function $F: I \to \mathbb{R}$ s.t. $F' = f$.
And we introduce the notation $F(x) = \int f(x)dx$

· Remark: Let $F, G: I \to \mathbb{R}$ be s.t. $F' = G' \Rightarrow F - G = \text{const.}$

$\int f(x)dx = F(x) + c$

Riemann-integration (Definite integration).



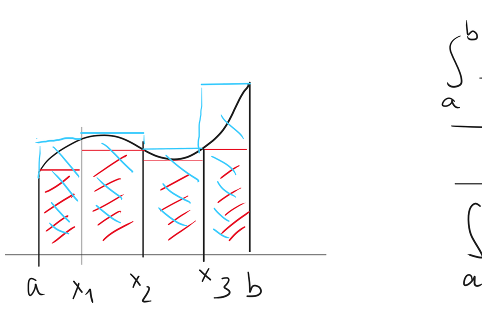
$f: [a,b] \to \mathbb{R}$ is a function.

Given an interval $I = [a,b]$, with $a, b \in \mathbb{R}$, $a < b$, we say that $P \subseteq [a,b]$ is a partition of $[a,b]$ if P is finite and $a, b \in P$.

$P = \{a = x_0, x_1, \dots, x_n = b\}$, $\quad a = x_0 < x_1 < \dots < x_n = b$

given $f: [a,b] \to \mathbb{R}$, we set

$$L(f,P) := \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x)\,(x_i - x_{i-1}) \quad \leq \quad U(f,P) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x)\,(x_i - x_{i-1})$$



$$\underline{\int_a^b} f(x)dx := \sup\{L(f,P) : P \subset [a,b] \text{ partition}\}$$

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(f,P) : P \subset [a,b] \text{ partitions}\}$$

$P_1 \subset P_2 \Rightarrow L(f,P_1) \leq L(f,P_2) \leq U(f,P_2) \leq U(f,P_1)$

$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$$

Definition (Riemann-integrable function). We say that a function $f: [a,b] \to \mathbb{R}$ is Riemann integrable if $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx =: \alpha \in \mathbb{R}$.

In particular, if $f: [a,b] \to \mathbb{R}$ is R-integrable, we set $\alpha := \int_a^b f(x)dx$

Examples:

Let $f: [0,1] \to \mathbb{R}$, $\quad f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ 0 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$

This function is not R-integrable.

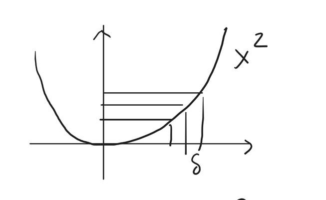
$$\underline{\int_0^1} f(x)dx = 0 < \overline{\int_0^1} f(x)dx = 1.$$

Uniform continuity:

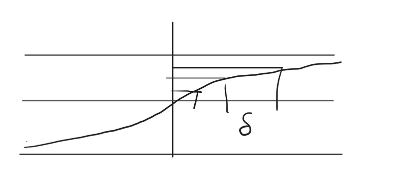
Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ be a function. We say that f is uniformly continuous on X if:
$\forall \varepsilon > 0, \exists \delta > 0$ ($\delta = \delta(\varepsilon)$): $\forall x, y \in X$, $|x-y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

Definition of continuity: f is cont in X if f is cont at x, $\forall x \in X$.
f is cont at $x \in X$ if $\forall \varepsilon > 0, \exists \delta > 0$: $\forall y \in X$, $|x-y| < \delta$, we $|f(x) - f(y)| < \varepsilon$.
$\delta = \delta(\varepsilon, x)$

Examples:



no u.c. on $\mathbb{R}$



u.c. on $\mathbb{R}$

Lagrange th: $f: \mathbb{R} \to \mathbb{R}$ differentiable. $\forall x, y \in \mathbb{R}$, $y > x$, $\exists\, \xi \in (x,y)$: $f(x) - f(y) = f'(\xi)(y-x)$

f' bounded $\Rightarrow$ f u.c.

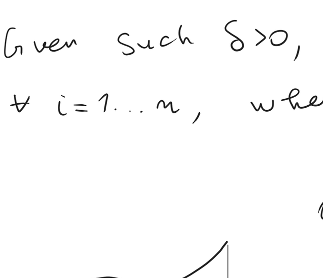
take $\varepsilon > 0$, $x, y \in \mathbb{R}$

$$|f(y) - f(x)| \leq |f'(\xi)|\,|y-x| \leq \sup_{\mathbb{R}} |f'|\,|x-y| < \varepsilon \quad \text{if } |x-y| < \delta := \frac{\varepsilon}{\sup_{\mathbb{R}} |f'|}$$

Thm: If $f: [a,b] \to \mathbb{R}$ cont, $a, b \in \mathbb{R}$, $a < b \Rightarrow f$ is uniformly continuous.

Thm (Integrabily of continuous functions)
Let $f: [a,b] \to \mathbb{R}$ be a continuous function, with $a, b \in \mathbb{R}$, $a < b$. Then f is R-integrable on $[a,b]$.

proof: f is uniformly cont.
$\forall \varepsilon > 0, \exists \delta > 0$: $\forall x, y \in [a,b]$, $|x-y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$
Given such $\delta > 0$, we consider a partition $P \subset [a,b]$ of $[a,b]$ s.t. $0 < x_i - x_{i-1} < \delta$, $\forall i = 1 \dots n$, where we have set $P = \{a = x_0, x_1, \dots, x_n = b\}$. $\quad a = x_0 < x_1 < \dots < x_n = b$



$$0 \leq U(f,P) - L(f,P) = \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x)\right)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \left(f(\xi_i) - f(\varsigma_i)\right)(x_i - x_{i-1}) \leq \varepsilon(b-a)$$

f cont $\Rightarrow \exists\, \xi_i, \varsigma_i \in [x_{i-1}, x_i]$: $\sup_{[x_{i-1},x_i]} f = f(\xi_i)$ and $\inf_{[x_{i-1},x_i]} f = f(\varsigma_i)$;
$\xi_i, \varsigma_i \in [x_{i-1}, x_i] \Rightarrow |\varsigma_i - \xi_i| < \delta$, $\quad \forall i = 1 \dots n$

$\Rightarrow |f(\varsigma_i) - f(\xi_i)| < \varepsilon$

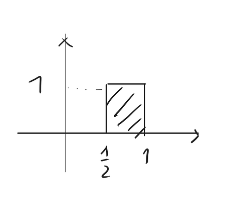
$$0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)dx \underset{\text{def of inf and sup}}{\leq} U(f,P) - L(f,P) \leq \varepsilon(b-a) \Rightarrow \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx =: \alpha$$

$$P = \{a,b\} \Rightarrow \inf_{[a,b]} f\,(b-a) \leq \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx \leq \sup_{[a,b]} f \cdot (b-a)$$

f cont on $[a,b] \Rightarrow f$ is bounded. $\Rightarrow \alpha \in \mathbb{R} \Rightarrow f$ is R-integrable. ☒

Is the opposite implication true? If: $f: [a,b] \to \mathbb{R}$ is R-integrable $\Rightarrow$ Is f continuous?
No

$f: [0,1] \to \mathbb{R}$, $\quad f(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}) \\ 1 & x \in [\frac{1}{2}, 1] \end{cases}$

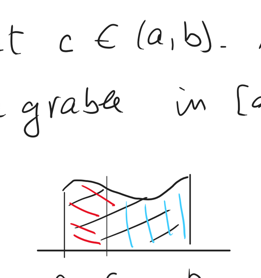


$$\int_0^1 f(x)dx = \frac{1}{2}$$

Theorem:

(i) (Additivity). Let $f: [a,b] \to \mathbb{R}$, $a < b$, $a, b \in \mathbb{R}$. Let $c \in (a,b)$. Assume that f is integrable in $[a,c]$ and in $[c,b]$. Then f is integrable in $[a,b]$ and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx.$$



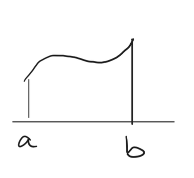
(ii) (Linearity). Let $f: [a,b] \to \mathbb{R}$, $g: [a,b] \to \mathbb{R}$ be R-integrable functions. Let $\alpha, \beta \in \mathbb{R}$.
Then $\alpha f + \beta g$ is R-integrable in $[a,b]$ and

$$\int_a^b (\alpha f(x)dx + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

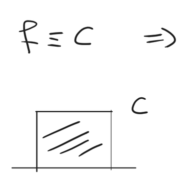
Theorem (Mean value theorem for integrals)
Let $f: [a,b] \to \mathbb{R}$ be a continuous function. Then $\exists\, c \in [a,b]$:

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx$$

Comment:



$$f \equiv c \Rightarrow \int_a^b f(x)dx = c(b-a) \Rightarrow \frac{1}{b-a} \int_a^b f(x)dx = c$$



proof: $P = \{a,b\}$, $\quad L(f,P) = \underbrace{\inf_{x \in [a,b]} f(x)}_{\in \mathbb{R} \text{ (f bounded)}}(b-a)$, $\quad U(f,P) = \underbrace{\sup_{x \in [a,b]} f(x)}_{\in \mathbb{R} \text{ (f bounded)}}(b-a)$

$$(b-a) \inf_{x \in [a,b]} f(x) = L(f,P) \leq \int_a^b f(x)dx \leq U(f,P) = \max_{x \in [a,b]} f(x)\,(b-a)$$

$$\min_{x \in [a,b]} f(x) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \max_{x \in [a,b]} f(x) \overset{f \text{ cont}}{\Rightarrow} \exists\, c \in [a,b]: f(c) = \frac{1}{b-a} \int_a^b f(x)dx \quad ☒$$

Theorem (Existence of a primitive). Let $f: [a,b] \to \mathbb{R}$ be a continuous function. with $a, b \in \mathbb{R}$, $a < b$. Then the function

$$F(x) := \int_a^x f(t)dt$$

is a primitive of f.

proof: Let $x \in (a,b)$. Let $\varepsilon > 0$. $\exists\, \delta > 0$: $\forall h \in (-\delta, \delta)$, $|f(x+h) - f(x)| < \varepsilon$.
Take $h \in (-\delta, \delta) \setminus \{0\}$,

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| = \left|\frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) - f(x)\right| \overset{\text{additivity}}{=} \left|\frac{1}{h} \int_x^{x+h} f(t)dt - f(x)\right|$$

$$= |f(\xi) - f(x)| < \varepsilon$$

m.v.t.: $\exists\, \xi \in [x, x+h]$: $\frac{1}{h} \int_x^{x+h} f(t)dt = f(\xi)$

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x) \quad ☒$$

Def: $F: [a,b] \to \mathbb{R}$ is known as the integral function of f.