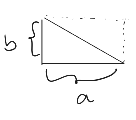
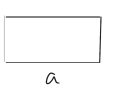
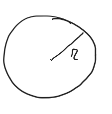
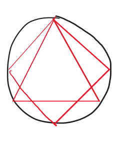
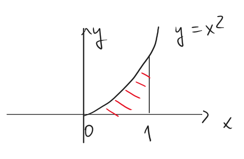
Integrals:

 Area $= \frac{b \cdot a}{2}$;  Area $= a \cdot b$;  Area $= \pi r^2$.

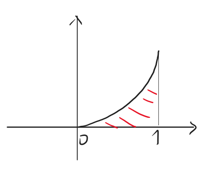




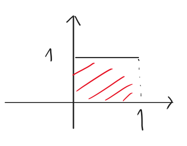
- Differentiation: $f \rightsquigarrow f'$. $f(x) = x^2 \rightsquigarrow f'(x) = 2x$.
- Indefinite integration: "inverse of differentiation".
  $g \rightsquigarrow$ we look for a function f whose derivative is g.
- Indefinite integration $\xleftrightarrow[\text{bridge}]{}$ Computation of the area.

bridge = Riemann integration (definite integration).

<u>Example:</u> We want to compute the area of the region between the x-axis and the parabola of equation $y = x^2$ in the interval $x \in [0,1]$



$$\text{Area} = \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$



$$1 = \text{Area} = \int_0^1 1\, dx = x\Big]_0^1 = 1 - 0$$

<u>Definition</u> (Primitive of a function) Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ be a function, X = open. We say that $F: X \to \mathbb{R}$ is a primitive of f if F is differentiable and $F' = f$.

<u>Proposition:</u> Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ be a function on an interval. (X = interval) Let $F_1, F_2: X \to \mathbb{R}$ be primitives of $f \Rightarrow F_1 = F_2 + c$, for some $c \in \mathbb{R}$.

proof: $(F_1 - F_2)' = F_1' - F_2' = f - f = 0 \Rightarrow F_1 - F_2 = c$, for some $c \in \mathbb{R}$.

$X = (0,1) \cup (2,3)$, $f' = 0$ on $X \not\Rightarrow f$ is constant

$$f = \begin{cases} 1 & \text{in } (0,1) \\ 2 & \text{in } (2,3) \end{cases}$$

<u>Definition</u> (Indefinite integral) Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ and $F: X \subseteq \mathbb{R} \to \mathbb{R}$ be functions. Assume that F is a primitive of f. Then we write

$F(x) = \int f(x)\,dx$

and we say that $F(x)$ is the indefinite integral of f.

<u>Remark:</u> If $\int f(x)\,dx = F(x) \Rightarrow \int f(x)\,dx = F(x) + c$, $\forall c \in \mathbb{R}$.

<u>Examples:</u>

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c, \quad \forall \alpha \in \mathbb{R} \setminus \{-1\}, \quad \forall x \in D(x^\alpha).$$

$\Rightarrow \int 1\, dx = x + c$, $\forall x \in \mathbb{R}$. $[1 = x^0]$

$\int \frac{dx}{x} = \log x + c \quad \forall x > 0$.

$\int e^x dx = e^x + c \quad \forall x \in \mathbb{R}$

$\int \sin x\, dx = -\cos x + c \quad \forall x \in \mathbb{R}$.

$\int \cos x\, dx = \sin x + c \quad \forall x \in \mathbb{R}$

$\int \frac{1}{\cos^2 x} dx = \tan x + c \quad \forall x \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x}{\cos x} + \sin x \frac{-1}{(\cos x)^2}(-\sin x) = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 y} = \frac{1}{\cos^2 x}$$

$\int \frac{1}{\sin^2 x} dx = -\cot x + c \quad \forall x \in \mathbb{R} \setminus \{\pi k : k \in \mathbb{R}\}$

$\int \frac{1}{1+x^2} dx = \arctan x + c \quad \forall x \in \mathbb{R}$.

$$(\arctan x)' = \frac{1}{\tan'(\arctan x)} = \frac{1}{1 + \tan^2(\arctan x)} = \frac{1}{1+x^2} \quad \forall x \in \mathbb{R}.$$

$\int \frac{-1}{1+x^2} dx = \operatorname{arccot} x + c \quad \forall x \in \mathbb{R}$.

$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c \quad \forall x \in (-1,1)$

$\int \frac{-1}{\sqrt{1-x^2}} dx = \arccos x + c \quad \forall x \in (-1,1)$

$$(\arcsin x)' = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}} \quad \forall x \in (-1,1)$$

$\sin^2\theta + \cos^2\theta = 1, \quad \forall \theta \in \mathbb{R}$

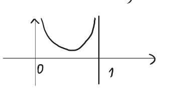
$\cos\theta = \sqrt{1 - \sin^2\theta}$

<u>Proposition</u> (Linearity). Let $f, g: I \subseteq \mathbb{R} \to \mathbb{R}$ be functions on an interval. Let $\alpha, \beta \in \mathbb{R}$. Then $\int (\alpha f(x) + \beta g(x))\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$.

$$\int (x + x^2)\,dx = \int x\,dx + \int x^2 dx = \frac{x^2}{2} + \frac{x^3}{3} + c$$

$$\int \frac{x^2+1}{x} dx = \int \left(\frac{x^2}{x} + \frac{1}{x}\right) dx = \int \left(x + \frac{1}{x}\right) dx = \frac{x^2}{2} + \log x$$

<u>Theorem</u> (Integration by parts). Let $f, g: I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable functions in $\mathring{I}$, where I is an interval. Assume that f, g are continuous in I.



Then $\int f g'\, dx = f g - \int f' g\, dx + c$

<u>Example:</u>

$$\int \log x\, dx = \int \underbrace{\log x}_{f} \cdot \underbrace{1}_{g'}\, dx = \underbrace{x}_{g} \cdot \underbrace{\log x}_{f} - \int \frac{1}{\cancel{x}} \cdot \cancel{x}\, dx = x \log x - \int 1\, dx = x \log x - x = x(\log x - 1) + c$$

<u>Theorem</u> (Change of variables)

Let $f: [a,b] \to \mathbb{R}$, $g: [c,d] \to [a,b]$ be continuous functions. Assume that g is differentiable in (a,b). Then

$$\int f \circ g(x)\, g'(x)\,dx = F \circ g(x) + c \quad \forall x \in [c,d], \text{ where } F(x) = \int f(x)\,dx$$

<u>proof:</u>

$$(F \circ g)'(x) = \underbrace{F' \circ g(x) \cdot g'(x)}_{\text{chain rule}} = f \circ g(x) \cdot g'(x).$$

<u>Example:</u> $\int \frac{f'(x)}{f(x)} dx = \log(f(x)) + c$

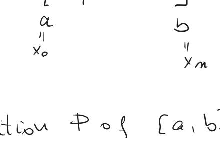
$(\log f(x))' = \frac{1}{f(x)} f'(x)$.

$\int \frac{2x}{x^2+1} dx = \log(x^2+1) + c$

<u>Riemann integrals:</u> $f: [a,b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$.

<u>Definition</u> (Partition of $[a,b]$). A partition P of $[a,b]$ is any finite set $P \subset [a,b]$ such that $a, b \in P$.

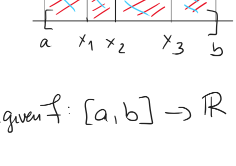
- So, given a partition P of $[a,b]$, we denote its points by $a = x_0 < x_1 < \cdots < x_n = b$.
- Given two partitions, P_1 and P_2 of $[a,b]$, we say that P_1 is finer than P_2 if $P_2 \subset P_1$



- Given a partition P of $[a,b]$, we define the lower sums and the upper sums of f relative to P as follows:

$$L(f,P) := \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x)\,(x_i - x_{i-1}) \leq U(f,P) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x)\,(x_i - x_{i-1})$$

<u>Remark:</u> assume that P_1 is finer than P_2



$\Rightarrow L(f, P_1) \geq L(f, P_2)$

and $U(f, P_1) \leq U(f, P_2)$.

<u>Def:</u> given $f: [a,b] \to \mathbb{R}$, we set

$$\underline{\int_a^b} f(x)\,dx := \sup\{L(f,P) : P \text{ is partition of } [a,b]\}$$

$$\overline{\int_a^b} f(x)\,dx := \inf\{U(f,P) : P \text{ is partition of } [a,b]\}$$

<u>Remark:</u> $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$.

<u>Definition</u> (Riemann integrable function). We say that a function $f: [a,b] \to \mathbb{R}$ is Riemann-integrable on $[a,b]$ if $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx =: \alpha \in \mathbb{R}$.

In particular, if it is the case, we set $\int_a^b f(x)\,dx = \alpha$.

<u>Example of non integrable function:</u>

$$f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ 0 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

$\underline{\int_0^1} f(x)\,dx = 0$ (because $L(f,P) = 0$, $\forall$ partition P of $[0,1]$)

$\overline{\int_0^1} f(x)\,dx = 1 \neq 0 = \underline{\int_0^1} f(x)\,dx$

(<u>Lebesgue integrals</u>: $L\int_0^1 f(x)\,dx = 0$)