Exercises Sheet 2

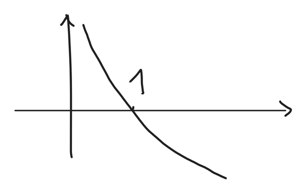
(3)

(d) $e^{x^2} - 4 \leq 0 \iff e^{x^2} \leq 4 \iff \ln e^{x^2} \leq \ln 4 \iff x^2 \leq \log 4 \iff -\sqrt{\log 4} \leq x \leq \sqrt{\log 4}$

$[\, x^2 - a \leq 0 \iff -\sqrt{a} \leq x \leq \sqrt{a},\ \forall a \geq 0 \,]$ $\sqrt{\log 4} = (\log 4)^{\frac{1}{2}} = (\log(2^2))^{\frac{1}{2}} = (2 \log 2)^{\frac{1}{2}} = \sqrt{2}\sqrt{\log 2}$

$\left(\frac{1}{2}\right)^{x^2} - 4 \leq 0 \iff \left(\frac{1}{2}\right)^{x^2} \leq 4 \iff \log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{x^2} \geq \log_{\frac{1}{2}} 4 \iff x^2 \geq \log_{\frac{1}{2}} 4$ true $\forall x \in \mathbb{R}$.

$0 < \frac{1}{2} < 1$ $\log_{\frac{1}{2}} 4 < 0$



$\left(\frac{1}{2}\right)^{x^2} - \frac{1}{4} \leq 0 \iff \left(\frac{1}{2}\right)^{x^2} \leq \frac{1}{4} \iff \log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{x^2} \leq \log_{\frac{1}{2}}\left(\frac{1}{4}\right) \iff x^2 \leq \log_{\frac{1}{2}}\left(\frac{1}{2}\right)^2 = 2$

$\iff x^2 \leq 2 \iff -\sqrt{2} \leq x \leq \sqrt{2}$.

(f) $\log_{\frac{1}{3}}\left(\frac{x^2-x-2}{x+9}\right) \geq 1 \iff \left(\left(\frac{1}{3}\right)^{\log_{\frac{1}{3}}\left(\frac{x^2-x-2}{x+9}\right)} \leq \left(\frac{1}{3}\right)^1 \quad \left(\frac{1}{3} < 1\right)\right) \wedge \frac{x^2-x-2}{x+9} > 0$

- $\frac{x^2-x-2}{x+9} \leq \frac{1}{3} \iff \frac{x^2-x-2}{x+9} - \frac{1}{3} \leq 0 \iff \frac{3(x^2-x-2)+(x+9)(-1)}{3(x+9)} \leq 0$

$\left[\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd} \quad \forall a,b,c,d \in \mathbb{R}, b \neq 0, d \neq 0\right]$

$\iff \frac{3x^2 - 3x - 6 - x - 9}{3(x+9)} \leq 0 \iff \frac{3x^2 - 4x - 15}{3(x+9)} \leq 0$

$3x^2 - 4x - 15 \leq 0$

$a = 3$
$b = -4$
$c = -15$

$\frac{4 \pm \sqrt{(-4)^2 - 4 \cdot (3)(-15)}}{6} = \frac{4 \pm 14}{6} = \frac{18}{6} = 3 \qquad \frac{-10}{6} = \frac{-5}{3}$

$3 \qquad \frac{-5}{3}$

$3x + 27 \leq 0$

$x = \frac{27}{3} = \quad x = 9 \qquad x = -9$

$(-\infty, -9) \cup \left[\frac{-5}{3}, 3\right]$

-9		$\frac{-5}{3}$		3	
+	+	−	+		
−	+	+	+		$x+9$
−	+	−	+		

- $\frac{x^2-x-2}{x+9} > 0$ $[\, a \in (0,1),\ \log_a z \geq y \iff 0 < z \leq a^y \,]$

- $N(x) = x^2 - x - 2 > 0$, $x_\pm = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \frac{4}{2} / -\frac{2}{2} = 2 / -1$

$a = 1, b = -1, c = -2$

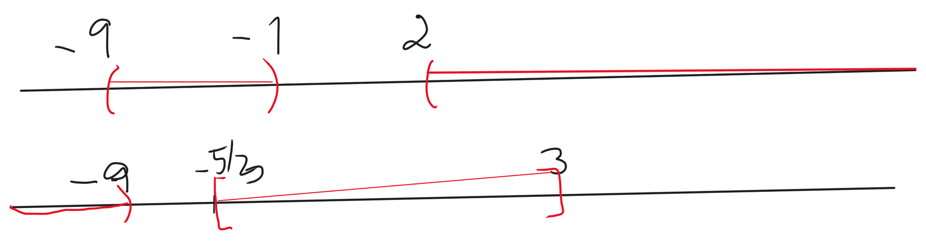
$N(x) = (x-2)(x+1)$

- $D(x) = x + 9 > 0 \iff x > -9$

		-9	-1	2
$x-2>0$	−	−	−	+
$x+1>0$	−	−	+	+
$x>-9 \iff x+9>0$	−	+	+	+
	−	+	−	+

- $\frac{x^2-x-2}{x+9} > 0 \iff x \in (-9, -1) \cup (2, +\infty)$

the final solution is $x \in \left((-9,-1) \cup (2,+\infty)\right) \cap \left((-\infty,-9) \cup \left[-\frac{5}{3}, 3\right]\right) = \left[-\frac{5}{3}, -1\right) \cup (2, 3]$



(g) $(\log_{\frac{1}{3}} x)^2 - 2\log_{\frac{1}{3}} x - 3 \geq 0 \iff y^2 - 2y - 3 \geq 0 \quad \wedge \quad x > 0$

$y = \log_{\frac{1}{3}} x$

$y^2 - 2y - 3 \geq 0 \qquad y_\pm = \frac{-\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - ac}}{a} = \frac{1 \pm \sqrt{1+3}}{1} = 1 \pm 2 = -1/3$

$y^2 - 2y - 3 = 0 \iff y = -1 \vee y = 3$

$y^2 - 2y - 3 \geq 0 \iff y \leq -1 \vee y \geq 3$

$y^2 - 2y - 3 = (y-3)(y+1) \geq 0$

	-1	3	
−	−	+	$y - 3 \geq 0 \iff y \geq 3$
−	+	+	$y + 1 \geq 0 \iff y \geq -1$
+	−	+	

$[\, ay^2 + bx + c = a(x - x_-)(x - x_+) \,]$

$\log_{\frac{1}{3}} x \leq -1 \vee \log_{\frac{1}{3}} x \geq 3 \iff x \geq \left(\frac{1}{3}\right)^{-1} = 3 \quad \vee \quad 0 < x \leq \left(\frac{1}{3}\right)^3 = \frac{1}{27}$

$\iff x \geq 3 \quad \vee \quad 0 < x \leq \frac{1}{27}$