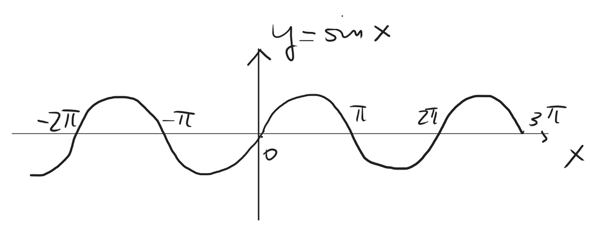
Tangent and cotangent:

We saw yesterday that $\sin(0) = \sin(\pi) = 0$, and the sine is a periodic function of period 2π. $\Rightarrow$ $\sin(2\pi k) = \sin(0) = 0 \quad \forall k \in \mathbb{Z}$

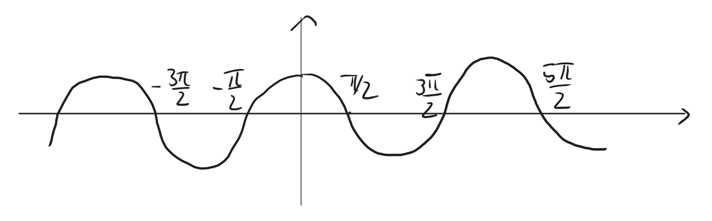
$\sin(\pi + 2\pi k) = \sin(\pi) = 0 \quad \forall k \in \mathbb{Z} \iff \sin(\pi(1+2k)) = 0 \quad \forall k \in \mathbb{Z}$.

$\sin x \neq 0 \quad \forall x \in (0,\pi) \cup (\pi, 2\pi)$. As a consequence, equation $\sin x = 0$ has infinitely many solutions, which are given by $\{k\pi\}_{k\in\mathbb{Z}}$. Or equivalently, we have

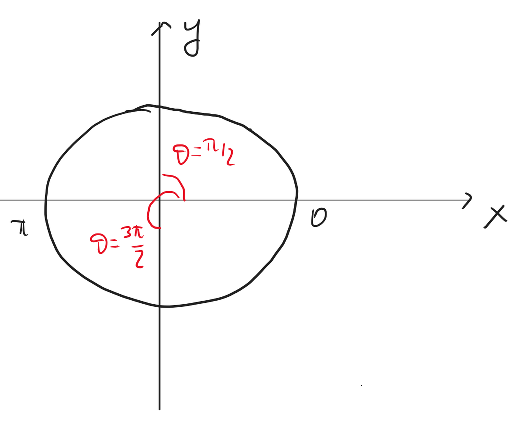
$\{x \in \mathbb{R} : \sin x = 0\} = \{k\pi : k \in \mathbb{Z}\}$



. we know that $\cos\frac{\pi}{2} = \cos\frac{3\pi}{3} = 0$ and $\cos x \neq 0$, $\forall x \in (0,2\pi) \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. Moreover, we know that the cosine is periodic of period 2π. $\Rightarrow$ the solutions to equation $\cos x = 0$ are $x = \frac{\pi}{2} + k\pi$, $\forall k \in \mathbb{Z}$. Equivalently, we have $\{x \in \mathbb{R} : \cos x = 0\} = \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$.



How to remember how to solve $\sin\theta = 0$ or $\cos\theta = 0$



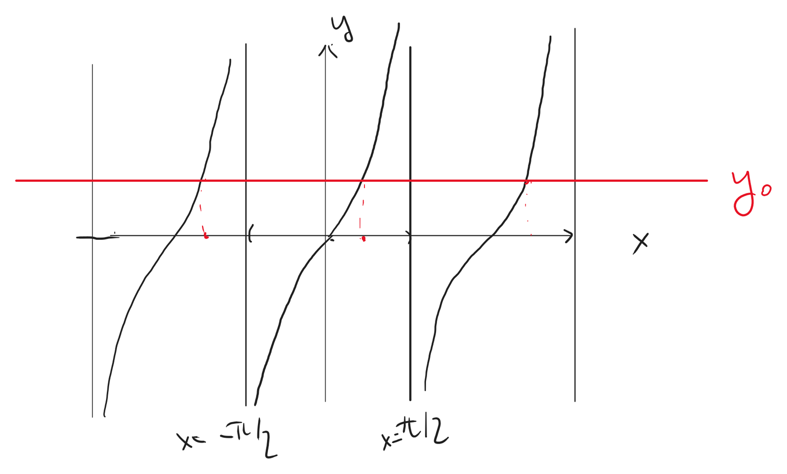
Definition: We define the function $\tan : \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\} \to \mathbb{R}$ by setting

$\tan\theta := \frac{\sin\theta}{\cos\theta}$

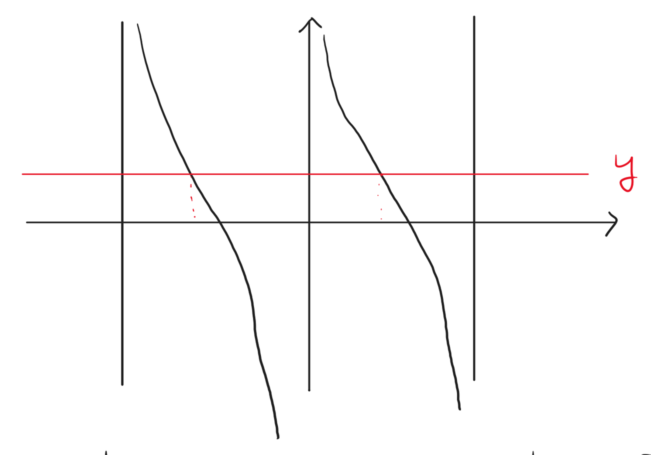
. We define the function $\cot : \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\} \to \mathbb{R}$ by setting

$\cot\theta := \frac{\cos\theta}{\sin\theta}$

Graphs.



the graph of the tangent



the graph of the cotangent.

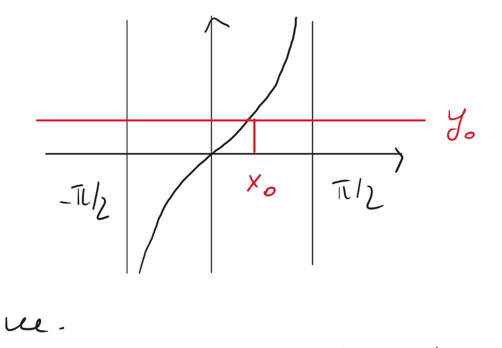
Remark: . the tangent and the cotangent are surjective but not injective.

. $\tan(\theta + \pi) = \tan\theta$, $\forall \theta \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$

$\cot(\theta + \pi) = \cot\theta$, $\forall \theta \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$

. In particular the restrictions of tan and cot to $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $(0,\pi)$ are bijective

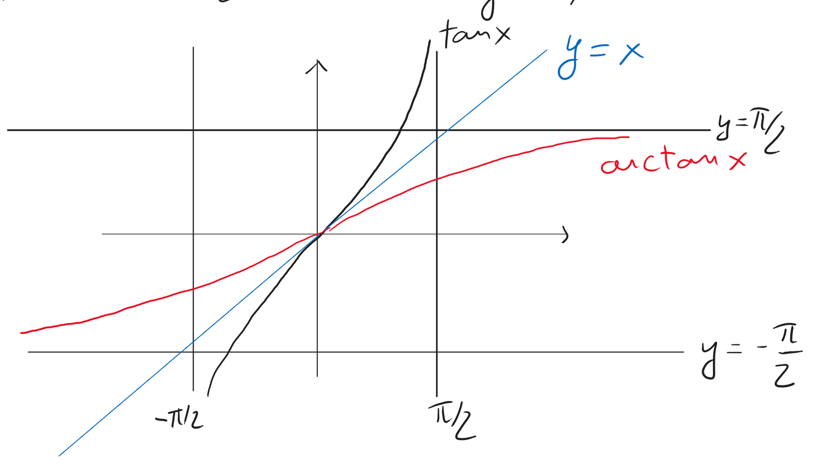
Let us consider the restriction $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$



- surjectivity: $\forall y_0 \in \mathbb{R}$, $\exists x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$: $\tan x_0 = y_0$
- injectivity: the point $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ found above is unique.

because "different inputs give different output"! Because tan is strictly increasing in $(-\frac{\pi}{2}, \frac{\pi}{2})$

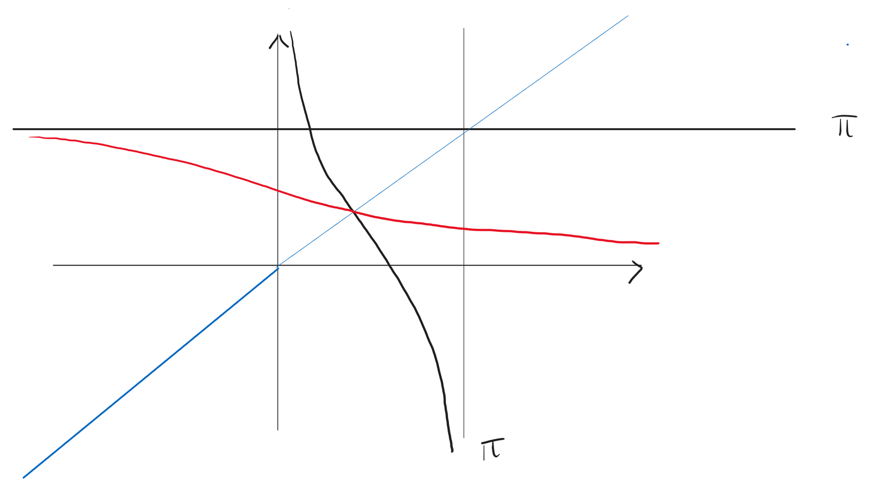
In particular the function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is invertible. The inverse is known as the arctangent, and denoted by $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$.



the red line is the graph of the arctangent.

$\arctan : \mathbb{R} \to \mathbb{R}$ is not surjective

$\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is surjective.



the red line is the graph of the arc cot $= (\cot)^{-1}$

Exercises:

Sheet 2, ex 6.

(a) Determine the domain of $f(x) = \log\left(\frac{x+2}{x^2}\right)$ and determine $f^{-1}(0,\infty)$.

$x \in D(f) \iff \frac{x+2}{x^2} > 0$: $N(x) = x+2 > 0 \iff x > -2$

$D(x) = x^2 > 0 \iff x \neq 0$

	-2		0	
N	$-$	$+$		$+$
D	$+$	$+$	0	$+$
	$-$	$+$		$+$

$\iff x \in (-2,0) \cup (0,+\infty)$

Answer: the domain of f is $D(f) = (-2,0) \cup (0,\infty)$.

$f^{-1}(0,\infty) = \{x \in D(f) : f(x) \in (0,\infty)\} = \{x \in D(f) : f(x) > 0\} = \{x \in D(f) : \log(\frac{x+2}{x^2}) > 0\} = (-1,0) \cup (0,2)$

we need to solve $\log\left(\frac{x+2}{x^2}\right) > 0$ and the solution to this inequality is the solution to the exercise. $[\mathrm{Log} = \log_{10}, \log = \ln = \log_e]$

$e^{\log\frac{x+2}{x^2}} > e^0 = 1 \quad (e > 1)$

$\frac{x+2}{x^2} > 1 \iff \frac{x+2-x^2}{x^2} > 0 \iff \frac{x^2-x-2}{x^2} < 0 \qquad x_\pm = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm 3}{2} = 2 / -1$

	-1		0		2	
	$-$	$+$		$+$		$+$
	$-$	$-$		$-$		$+$
	$+$	$+$	0	$+$		$+$
	$+$	$-$		$-$		$+$

$\boxed{x \in (-1,0) \cup (0,2) \subset D(f)}$