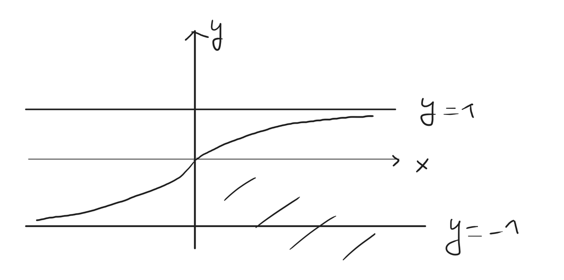
Sheet 6, ex 2,c)

Draw the graph of the function $f(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}$

(I) $D(f)=\{x\in\mathbb{R}: e^{-x}+e^x\neq 0\}=\mathbb{R}$; Symmetries: f is odd.

(II) Zeros of f: $e^x-e^{-x}=0 \iff e^x=e^{-x} \iff e^x=\frac{1}{e^x} \iff e^{2x}=1 \iff 2x=\log 1=0 \iff x=0$.

(III) the sign of f: $f(x)>0 \iff e^x-e^{-x}>0 \iff e^x>e^{-x}=\frac{1}{e^x} \iff e^{2x}>1 \iff 2x>\log 1=0 \iff x>0$.



(IV) $\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to\infty}\frac{\cancel{e^x}(1-e^{-2x})}{\cancel{e^x}(1+e^{-2x})}=1$

$\frac{x^2}{x}=x\to+\infty$ $\qquad \left[y=e^x, \quad \frac{y-\frac{1}{y}}{y+\frac{1}{y}}\right]$

$\{y=1\}$ is an asymptote.

(V) Critical points, monotonicity: $f(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}} \quad \forall x\in\mathbb{R}$

$$f'(x)=\frac{(e^x-e^{-x})'}{e^x+e^{-x}}+(e^x-e^{-x})\left(\frac{1}{e^x+e^{-x}}\right)'=\frac{e^x+e^{-x}}{e^x+e^{-x}}+(e^x-e^{-x})\frac{-(e^x-e^{-x})}{(e^x+e^{-x})^2}=1-\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2$$

$[(gh)'=g'h+gh']$ $\quad (e^{-x})'=e^{-x}(-x)'=-e^{-x}$ (chain rule) $\quad \Rightarrow \boxed{f'(x)=1-f(x)^2}$

$$((e^x+e^{-x})^{-1})'=(-1)(e^x+e^{-x})^{-2}(e^x+e^{-x})'=-(e^x+e^{-x})^{-2}(e^x-e^{-x})$$

$f'(x)=0 \iff 1-f(x)^2=0 \iff 1=f^2(x) \iff 1=\Big(\underbrace{\frac{e^x-e^{-x}}{e^x+e^{-x}}}_{y}\Big)^2 \iff 1=y^2 \iff y=1 \vee y=-1$

$\iff \frac{e^x-e^{-x}}{e^x+e^{-x}}=1 \ \vee\ \frac{e^x-e^{-x}}{e^x+e^{-x}}=-1 \iff \cancel{e^x}-e^{-x}=\cancel{e^x}+e^{-x} \ \vee\ e^x-e^{-x}=-(e^x+e^{-x})$

$\iff -e^{-x}=e^{-x} \ \vee\ e^x-\cancel{e^{-x}}=-e^x-\cancel{e^{-x}} \iff 2e^{-x}=0 \ \vee\ 2e^x=0$ No solution.

$\Rightarrow f$ has no critical points. $\Rightarrow f'(x)>0 \ \forall x\in\mathbb{R}$

· Thanks to symmetry, $\lim_{x\to-\infty}f(x)=-1$ and in particular $\{y=-1\}$ is a horizontal asymptote.

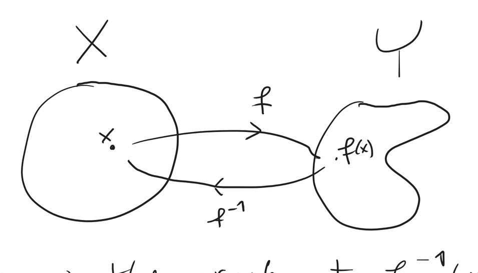
This function $f(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}$ is known as the <u>hyperbolic tangent</u> (tanh) and satisfies $(\tanh x)'=1-(\tanh x)^2>0 \quad \forall x\in\mathbb{R}$.

$\Rightarrow$ tanh is invertible from $\mathbb{R}$ to $(-1,1)$ and $\qquad \tanh:\mathbb{R}\xrightarrow{f}(-1,1)$, f^{-1}

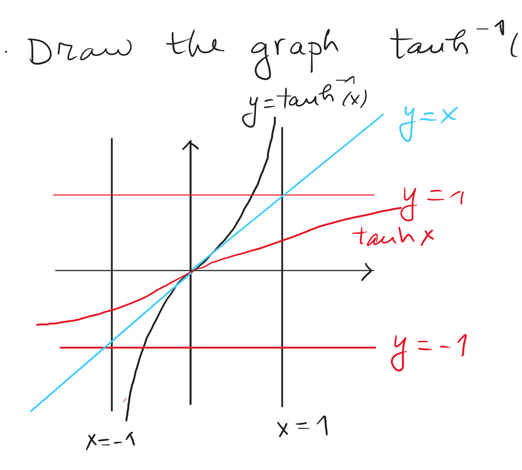
$$(\tanh^{-1})'(y)=\frac{1}{(\tanh)'((\tanh)^{-1}(y))}=\frac{1}{1-(\tanh((\tanh)^{-1}(y)))^2}=\frac{1}{1-y^2}$$

$$(f^{-1})'(y)=\frac{1}{f'(f^{-1}(y))}$$

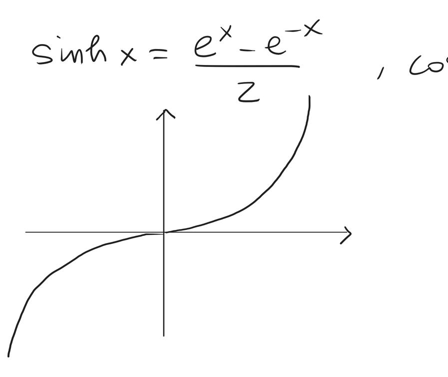
$\tanh^{-1}: (-1,1)\to\mathbb{R}$

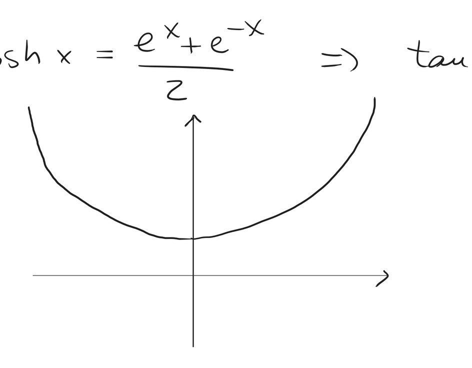


· Draw the graph $\tanh^{-1}(x)$



$\sinh x=\frac{e^x-e^{-x}}{2}$, $\cosh x=\frac{e^x+e^{-x}}{2}$ $\Rightarrow \tanh(x)=\frac{\sinh x}{\cosh x}$





· Where is tanh convex? (and concave)

$f(x)=\tanh x$

$f'(x)=1-f^2$

$$f''=-2ff'\begin{cases}>0 \iff f<0 \iff x<0\\ <0 \iff f>0 \iff x>0\\ =0 \iff f=0 \iff x=0\end{cases}$$

· f is convex in $(-\infty,0)$, concave in $(0,+\infty)$ and $x=0$ is the unique inflection point.

<u>Remark about symmetry:</u>

$X\subseteq\mathbb{R}$ is symmetric w.r. to 0. $f:X\to\mathbb{R}$ is an odd function (twice diff in $\mathring{X}$)

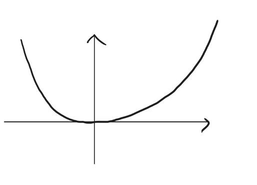
$\Rightarrow$ · $f(0)=0$. $\quad f(x)=-f(-x)$, $\forall x\in X$; $0\in X \Rightarrow f(0)=-f(0) \iff 2f(0)=0 \iff f(0)=0$.

· f' is even $\quad f(x)=-f(-x) \Rightarrow f'(x)=(-f(-x))'=-(f(-x))'=-(-f'(-x))=f'(-x)$.

· f'' is odd. $(\Rightarrow f''(0)=0)$.

f is even. $\Rightarrow$ · f' is odd

· f'' is even.



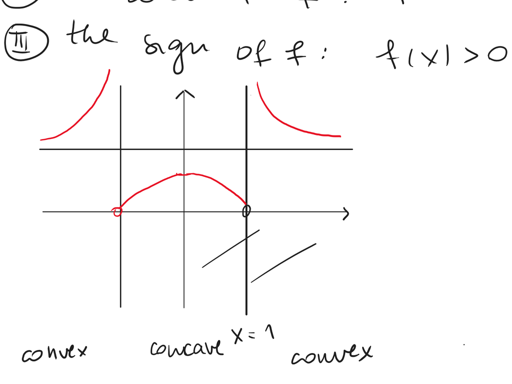
Sheet 6, ex 2,d)

$f(x)=e^{\frac{x^2+1}{x^2-1}}$

(I) $D(f)=\mathbb{R}\setminus\{\pm1\}$; f is even $[f(x)=f(-x)]$.

(II) the zeros of f: f has no zeros.

(III) the sign of f: $f(x)>0 \quad \forall x\in D(f)$.



(IV) limits.

· $\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}e^{\frac{\cancel{x^2}(1+(1/x^2))}{\cancel{x^2}(1-(1/x^2))}}=e$ $\quad (1/x^2\to0)$

· $\lim_{x\to1^+}f(x)=+\infty$

· $\lim_{x\to1^-}f(x)=0^+$

· Does $\lim_{x\to1}f(x)$ exist? <u>No</u>

(V) Monotonicity, critical points: $f(x)=e^{\frac{x^2+1}{x^2-1}}$

$$f'(x)=\left(\frac{x^2+1}{x^2-1}\right)'e^{\frac{x^2+1}{x^2-1}}=e^{\frac{x^2+1}{x^2-1}}\left(\frac{(x^2+1)'}{x^2-1}+((x^2-1)^{-1})'(x^2+1)\right)=e^{\frac{x^2+1}{x^2-1}}\left(\frac{2x}{x^2-1}-(1)(x^2-1)^{-2}2x(x^2+1)\right)$$

$$=e^{\frac{x^2+1}{x^2-1}}\left(\frac{2x}{x^2-1}-\frac{2x(x^2+1)}{(x^2-1)^2}\right)=e^{\frac{x^2+1}{x^2-1}}\frac{2x(x^2-1)-2x(x^2+1)}{(x^2-1)^2}=e^{\frac{x^2+1}{x^2-1}}\frac{2x}{(x^2-1)^2}(\cancel{x^2}-1-\cancel{x^2}-1)$$

$$f'(x)=e^{\frac{x^2+1}{x^2-1}}\frac{-4x}{(x^2-1)^2}<0 \qquad \forall x\in\mathbb{R}\setminus\{\pm1,0\}$$

$f'(x)=0 \iff x=0$.

· $x=0$ is the unique critical point of f. 0 is a strict local maximum.

· f is strictly decreasing in $(0,1)$ and in $(1,+\infty)$. f is strictly increasing in $(-1,0)$ and in $(-\infty,-1)$.

Sheet 6, ex 1,c)

$\lim_{x\to0}x\sin\frac{1}{x}=0 \qquad -x\le x\sin\frac{1}{x}\le x \qquad |\sin y|\le1 \quad \forall y\in\mathbb{R}$.

1,d) $\lim_{x\to1^+}(x^2-1)^{\frac{1}{x-1}}=+\infty$.

$$(x^2-1)^{\frac{1}{x-1}}=((x-1)(x+1))^{\frac{1}{x-1}}\sim(2(x-1))^{\frac{1}{x-1}}\sim2^{\frac{1}{x-1}}(x-1)^{\frac{1}{x-1}}\sim(x-1)^{\frac{1}{x-1}}$$

$$\lim_{y\to0^+}y^{1/y}=\lim_{y\to0^+}e^{\log(y^{1/y})}=\lim_{y\to0^+}e^{\frac{\log y}{y}}=\lim_{y\to0^+}e^{\frac{1}{y}}=+\infty.$$

$y\xmapsto{g}y^{1/y} \qquad D(g)=(0,\infty)$

<u>Example:</u>

$f(x)=1-\sqrt{\frac{x}{x+1}}$

(I) $D(f)=\{x\in\mathbb{R}: \frac{x}{x+1}>0\}=(-\infty,-1)\cup(0,+\infty)$

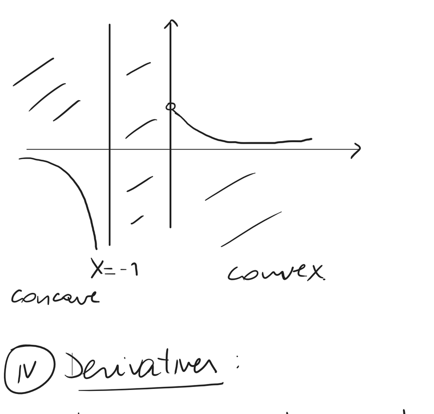
-1		0
−	−	+
−	+	+
+	−	+

(II) Zeros of f.

$1-\sqrt{\frac{x}{x+1}}=0 \iff \sqrt{\frac{x}{x+1}}=1 \iff \frac{x}{x+1}=1$ Never.

(III) the sign of f:

$1-\sqrt{\frac{x}{x+1}}>0 \iff \sqrt{\frac{x}{x+1}}<1 \iff \frac{x}{x+1}<1 \iff \frac{x-x-1}{x+1}<0 \iff \frac{1}{x+1}>0 \iff x>-1,\ x\in D(f)$



(V) $\lim_{x\to+\infty}f(x)=\lim_{x\to\infty}\left(1-\sqrt{\frac{x}{x+1}}\right)=0^+$

$\lim_{x\to-1^-}f(x)=-\infty$

$\lim_{x\to0^+}\left(1-\sqrt{\frac{x}{x+1}}\right)=1$

$\lim_{x\to-\infty}\left(1-\sqrt{\frac{x}{x+1}}\right)=0^-$

(IV) <u>Derivatives</u>:

$$f'(x)=-\frac{1}{2}\left(\frac{x}{x+1}\right)^{-1/2}\left(\frac{x}{x+1}\right)'=-\frac{1}{2}\sqrt{\frac{x+1}{x}}\cdot\left(\frac{1}{x+1}+x\frac{-1}{(x+1)^2}\cdot1\right)=-\frac{1}{2}\sqrt{\frac{x+1}{x}}\frac{\cancel{x}+1-\cancel{x}}{(x+1)^2}=-\frac{1}{2}\sqrt{\frac{x+1}{x}}\frac{1}{(x+1)^2}<0$$

· f is decreasing in $(-\infty,-1)$ and in $(0,+\infty)$.

f has no critical points.

$$f'(x)=-\frac{1}{2}\frac{1}{\sqrt{x}(x+1)^{3/2}}$$