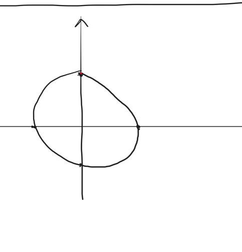
Sine and cosine of known angles:



- $\alpha = 0 \iff \theta = 0$. $p = (1,0) = (\cos\theta, \sin\theta)$

  $\Rightarrow \cos 0 = 1, \sin 0 = 1$

- $\alpha = 90° \iff \theta = \frac{\pi}{2}$, $p = (0,1) = (\cos\frac{\pi}{2}, \sin\frac{\pi}{2})$

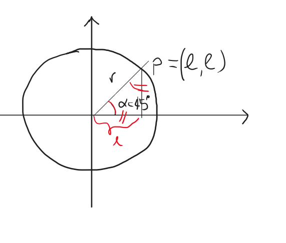
  $\Rightarrow \cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1$

- $\alpha = 180° \iff \theta = \pi$. $p = (-1,0) = (\cos\pi, \sin\pi)$

  $\cos\pi = -1, \sin\pi = 0$

- $\alpha = 270° \iff \theta = \frac{3}{2}\pi$. $p = (0,-1) = (\cos(\frac{3}{2}\pi), \sin(\frac{3\pi}{2}))$

- $\alpha = 360° \iff \theta = 2\pi$. $p = (1,0) = (\cos 2\pi, \sin 2\pi)$



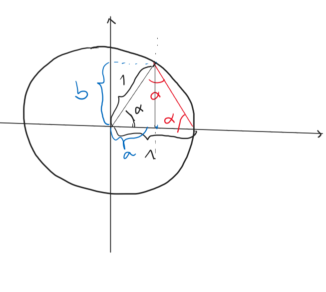
$\alpha = 45° \iff \theta = \frac{\pi}{4} = (\frac{1}{2}\cdot\frac{\pi}{2})$

$r = 1$

$\ell^2 + \ell^2 = r^2$ (Pitagora's theorem)

$r = 1 \Rightarrow \ell^2 + \ell^2 = 1 \iff 2\ell^2 = 1 \iff \ell^2 = \frac{1}{2}$

$\ell = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \ell = \cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$



$\alpha = 60° \iff \theta = \frac{\pi}{3}$; $a = \cos\theta$, $b = \sin\theta$

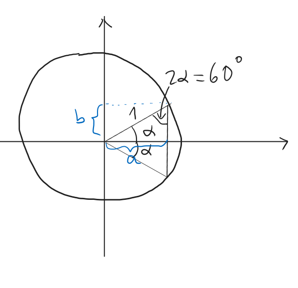
$a = \frac{1}{2} = \cos\frac{\pi}{3}$

Pitagora's theorem: $a^2 + b^2 = 1$

$\iff \left(\frac{1}{2}\right)^2 + b^2 = 1 \iff \frac{1}{4} + b^2 = 1 \iff b^2 = 1 - \frac{1}{4} = \frac{4-1}{4}$

$\iff b^2 = \frac{3}{4} \iff b = \pm\frac{\sqrt{3}}{2}$

$b = \sin\left(\frac{\pi}{3}\right) > 0 \Rightarrow b = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$



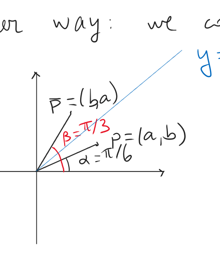
$\alpha = 30° \iff \theta = \frac{\pi}{6}$

$b = \frac{1}{2}$, a can be determined with the Pitagora theorem.

$a^2 + b^2 = 1 \iff a^2 + \left(\frac{1}{2}\right)^2 = 1 \iff a = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

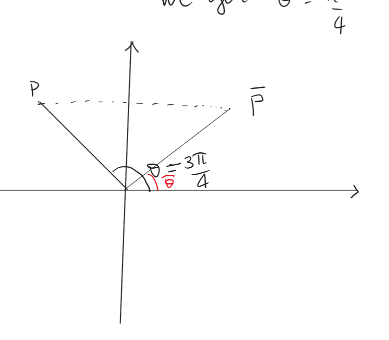
Another way: we can simply use reflection.



$\alpha = 0$	$\theta = 0$	$\cos\theta = 1$	$\sin\theta = 0$
30°	$\pi/6$	$\sqrt{3}/2$	$1/2$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
60°	$\pi/3$	$1/2$	$\sqrt{3}/2$
90°	$\pi/2$	0	1
		$\searrow$ in $(0, \frac{\pi}{2})$	$\nearrow$ in $(0, \frac{\pi}{2})$

$\cos\left(\frac{3\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$; $\sin\left(\frac{3\pi}{4}\right) =$

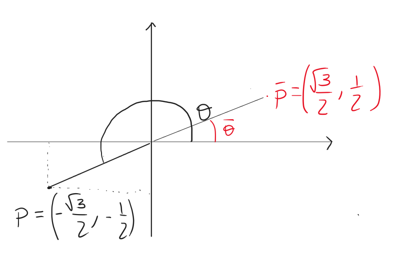
if we reflect with respect to the y axis, we get $\bar{\theta} = \frac{\pi}{4}$



$\bar{\theta} = \frac{\pi}{4} \Rightarrow \bar{p} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\Rightarrow p = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\Rightarrow \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}, \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

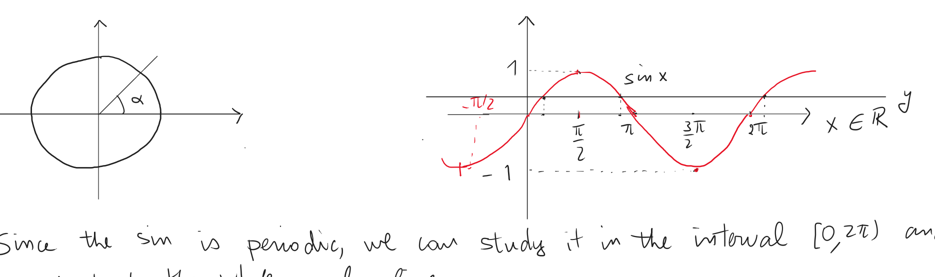


$\theta = \pi + \frac{\pi}{6} = \frac{7}{6}\pi$

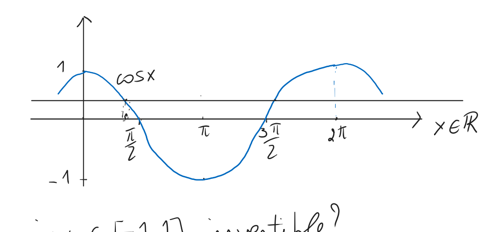
$\cos\left(\frac{7}{6}\pi\right) = -\frac{\sqrt{3}}{2}$

$\sin\left(\frac{7}{6}\pi\right) = -\frac{1}{2}$

The graphs of the sin and cos:



Since the sin is periodic, we can study it in the interval $[0, 2\pi)$ and then extend it to the whole real line.



<u>Question</u>: is the function $x \in \mathbb{R} \mapsto \sin x \in [-1,1]$ invertible?

- $\forall y \in [-1,1], \exists x \in \mathbb{R} : \sin x = y$. $\Rightarrow \sin : \mathbb{R} \to [-1,1]$ is surjective.
- the function $\sin : \mathbb{R} \to [-1,1]$ is not injective.

  $\Rightarrow$ it is not invertible.

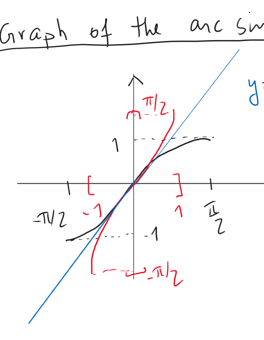
<u>Remark</u>: sin is increasing in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (strictly increasing)

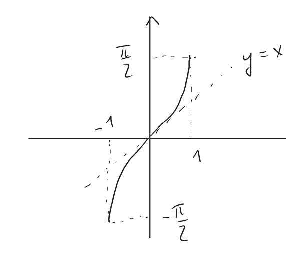
$\Rightarrow$ the restriction of sin to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is injective ["old" Lemma]

$\Rightarrow$ the restriction $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ is invertible with inverse

$\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

Graph of the arc sin:

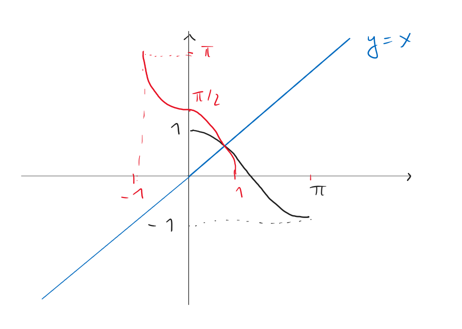




Inverse of the cosine:

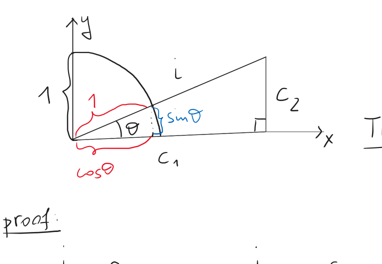
$\cos : x \in \mathbb{R} \mapsto \cos x \in [-1,1]$ is surjective but not injective.

The restriction $\cos : [0,\pi] \to [-1,1]$ is injective. $\Rightarrow$ the restriction $\cos : [0,\pi] \to [-1,1]$ is invertible. The inverse is known as the $\arccos : [-1,1] \to [0,\pi]$



<u>Remarks</u>: the function sin is odd, the cosine is even.

How to compute the length of the edges of a rectangle triangle.



i = ipotenuse

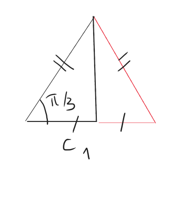
c_1 = cateto$_1$ (leg)

<u>Theorem</u> (sin theorem) $c_1 = i\cos\theta$

$c_2 = i\sin\theta$

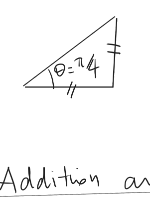
<u>proof</u>:

$\frac{i}{1} = \frac{c_1}{\cos\theta}$; $\frac{i}{1} = \frac{c_2}{\sin\theta}$



$\left.\begin{array}{l} i^2 = c_1^2 + c_2^2 \\ \boxed{c_1 = \frac{i}{2}} \end{array}\right\} \Rightarrow i^2 = \frac{i^2}{4} + c_2^2 \iff c_2^2 = \frac{3}{4}i^2 \iff \boxed{c_2 = \frac{\sqrt{3}}{2}i}$

Pitagora's theorem.



$\begin{array}{l} c_1 = c_2 \\ c_1^2 + c_2^2 = i^2 \end{array} \Rightarrow 2c_1^2 = i^2 \iff c_2 = c_1 = \frac{i}{\sqrt{2}} = \frac{\sqrt{2}}{2}i$

Addition and subtraction formulas:

- See my notes, section 5.1.1.

Duplication and bisection formulas

- See my notes, section 5.1.2

<u>Remark</u>: please, don't learn them by heart.

<u>Question</u>:

$\log_{\frac{1}{3}}\left(\frac{x^2-x-2}{x+2}\right) \geq 1$

$a = \frac{1}{3} < 1$. if $y \geq z \Rightarrow \left(\frac{1}{3}\right)^y \leq \left(\frac{1}{3}\right)^z$

<u>hint</u>: $\frac{1}{3}^{\log_{\frac{1}{3}}\left(\frac{x^2-x-2}{x+2}\right)} \leq \left(\frac{1}{3}\right)^1$

$a^{\log_a z} = z$

$\forall a \in (0,1) \cup (1,\infty)$

$\forall z > 0$

$\begin{cases} (1) & \frac{x^2-x-2}{x+2} \leq \frac{1}{3} \\ (2) & \frac{x^2-x-2}{x+2} > 0 \end{cases}$

$\begin{cases} \ldots\ldots \\ \ldots\ldots \end{cases}$ take the intersection