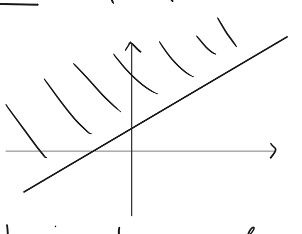
Convexity:

. A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is convex if its epigraph $\{(x,y) \in \mathbb{R}^2 : y > f(x)\}$ is convex.

. Assume that $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is twice differentiable. Then f is convex $\iff f''(x) \geq 0 \ \forall x \in I$.

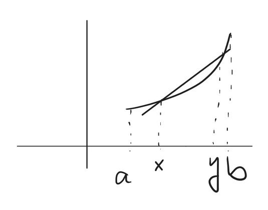
. If $f''(x) > 0 \ \forall x \in I \Rightarrow f$ is convex
$\not\Leftarrow$

Example: if $f(x) = ax+b$, $a,b \in \mathbb{R} \Rightarrow f$ is convex in $\mathbb{R}$ but $f'' = 0$



Characterization of convex functions:

Thm
Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a function on an interval $I \subseteq \mathbb{R}$. Assume that f is continuous. Then f is convex $\iff f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$, $\forall x,y \in I, \forall \lambda \in (0,1)$



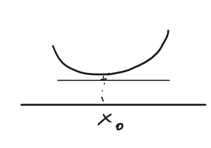
$I = [a,b]$

In other words a function is convex if and only if its graph lies below the secant line in 2 points, for any couple of points $(x, f(x))$, $(y, f(y))$.

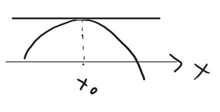
The relation between convexity and critical points.

Let $I \subseteq \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$ be a twice differentiable function. Let $x_0 \in \mathring{I}$ be a critical point. Then the following holds true

(i) if $f''(x_0) > 0 \Rightarrow x_0$ is a local minimizer



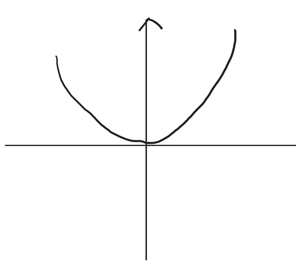
(ii) if $f''(x_0) < 0 \Rightarrow x_0$ is a local maximizer.



The opposite implications do not necessarily hold true.

Example: $f(x) = x^4$

$\inf_{x \in \mathbb{R}} f(x) = 0 = f(0)$



What $f''(0)$?

$f'(x) = 4x^{4-1} = 4x^3$; $f''(x) = 3 \cdot 4x^2 = 12x^2 \Rightarrow f''(0) = 0$

0 is a local minimizer but $f''(0) = 0$.

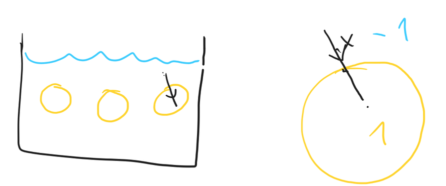
Definition: Let $I \subseteq \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$ be a twice differentiable function. Let $x_0 \in \mathring{I}$ be a critical point. We say that x_0 is nondegenerate if $f''(x_0) \neq 0$.

Remark: If x_0 is a non-degenerate, then x_0 is either a local minimizer or a local maximizer.

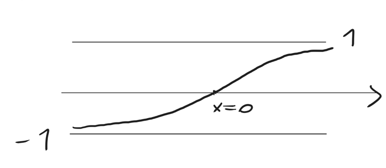
Definition: Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let $x_0 \in \mathring{X}$ and assume that f is twice differentiable at x_0. We say that x_0 is an inflection point if $f''(x_0) = 0$.

Remark: If x_0 is a saddle point $\Rightarrow x_0$ is an inflection point and f is twice differentiable at x_0

Exercises:



concentration(x) of oil $= \tanh(x/\sqrt{2}) = \dfrac{e^{x/\sqrt{2}} - e^{-x/\sqrt{2}}}{e^{x/\sqrt{2}} + e^{x/\sqrt{2}}}$



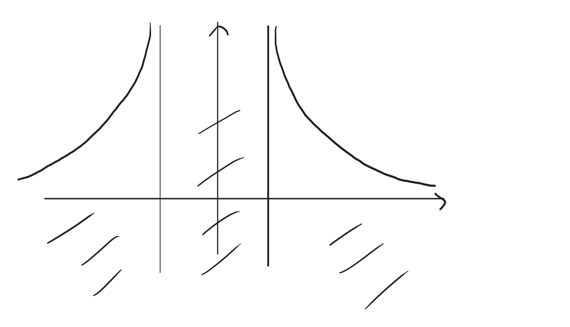
Exercises:

2,a) $f(x) = \log\left(\dfrac{x^2+1}{x^2-1}\right)$

(I) $D(f) = \left\{x \in \mathbb{R} : \dfrac{x^2+1}{x^2-1} > 0\right\} = (-\infty,-1) \cup (1,+\infty)$

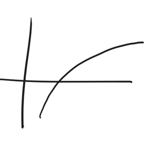
(II) Zeros of f: $f(x) = 0$ never. $\dfrac{x^2+1}{x^2-1} = 1 \iff \cancel{x^2}+1 = \cancel{x^2}-1$ never.
f never vanishes

(III) The sign of f: $f(x) > 0 \ \forall x \in D(f)$. $\left[\log\left(\dfrac{x^2+1}{x^2-1}\right) > 0 \iff \dfrac{x^2+1}{x^2-1} > 1 \iff x^2+1 > x^2-1, \forall x \in D(f)\right]$



(IV) $\lim_{x \to +\infty} f(x) = 0$; $\lim_{x \to 1^+} \log\left(\dfrac{2}{0^+}\right) = +\infty$

$\dfrac{1 \cdot x^2+1}{1 \cdot x^2-1} \to 1 \quad x \to +\infty$



(V) Monotonicity - critical points:

$$f'(x) = \frac{1}{\frac{x^2+1}{x^2-1}} \cdot \left(\frac{x^2+1}{x^2-1}\right)' = \frac{x^2-1}{x^2+1}\left(\frac{(x^2+1)'}{x^2-1} + (x^2+1)\left(\frac{1}{x^2-1}\right)'\right) =$$

$(\log x)' = \dfrac{1}{x}$

$$= \frac{x^2-1}{x^2+1}\left(\frac{2x}{x^2-1} + (x^2+1)(-1)(x^2-1)^{-2}2x\right)$$

$(fg)' = f'g + fg'$

$\left(\frac{1}{f}\right)' = (f^{-1})' = (-1)f^{-2} \cdot f'$

$$= \frac{x^2-1}{x^2+1}\left(\frac{2x}{x^2-1} - \frac{(x^2+1)2x}{(x^2-1)^2}\right)$$

$$= \frac{\cancel{x^2-1}}{x^2+1}(2x)\frac{\cancel{x^2}-1-(\cancel{x^2}+1)}{(x^2-1)^{\cancel{2}}} = \frac{2x}{(x^2+1)}\,\frac{-2}{(x^2-1)}$$

$$= -\frac{4x}{(x^2+1)(x^2-1)} < 0 \qquad \forall x > 1$$

. f is strictly decreasing in $(1,+\infty)$ and has no critical points.

(VI) Convexity

$$f''(x) = \left(-\frac{4x}{(x^2+1)(x^2-1)}\right)' = \frac{(-4x)'}{(x^2+1)(x^2-1)} - 4x\left(\frac{1}{(x^2+1)(x^2-1)}\right)' =$$

$$= \frac{-4}{(x^2+1)(x^2-1)} - 4x\left(\frac{1}{x^4-1}\right)' = \frac{-4}{x^4-1} - 4x(-1)(x^4-1)^{-2}(4x^3)$$

$$= \frac{-4}{x^4-1} + \frac{16x^4}{(x^4-1)^2} = \frac{-4(x^2-1)+16x^4}{(x^4-1)^2} = \frac{16x^4-4x^2+4}{(x^4-1)^2} = 4\,\frac{4x^4-x^2+1}{(x^2-1)^2} > 0 \quad \forall x \in D(f)$$

$4x^4 - x^2 + 1 > 0 \qquad x^2 = y$

$4y^2 - y + 1 > 0, \forall y \quad y_\pm = 1 \pm \cancel{\sqrt{1-16}} \qquad \Delta < 0$

$\Rightarrow f$ is convex in $(1,+\infty) \cup (-\infty,-1)$

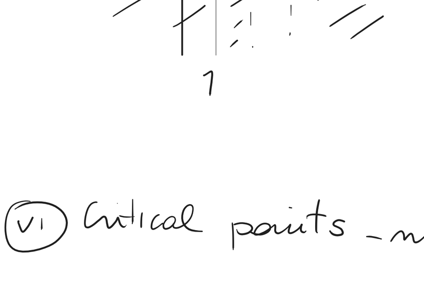
2, b) $f(x) = \dfrac{x^2-5x+6}{(x-1)^2}$

(I) $D(f) = \mathbb{R} \setminus \{1\}$

(II) Zeros of f. $x^2-5x+6 = 0 \qquad x_\pm = \dfrac{5 \pm \sqrt{25-24}}{2} = 2/3$

(III) the sign of f. $f(x) > 0$ if $x^2-5x+6 > 0 \wedge x \in D(f)$. $\iff x \in (-\infty,1) \cup (1,2) \cup (3,+\infty)$

	1	2	3	
−	−	+	+	$x-2$
−	−	−	+	$x-3$
+	+	+	+	$(x-1)^2$
+	+	−	+	



(V) $\lim_{x \to \pm\infty} \dfrac{x^2-5x+6}{(x-1)^2} = 1$

$\{x = 1\}$ is a vertical asymptote and $\{y = 1\}$ is a horizontal asymptote.

$\lim_{x \to 1^\pm} \dfrac{x^2-5x+6}{(x-1)^2} = \dfrac{2}{0^+} = +\infty$

(VI) Critical points - monotonicity:

$$f'(x) = \frac{2x-5}{(x-1)^2} + (x^2-5x+6)(-2)(x-1)^{-3} = \frac{2x-5}{(x-1)^2} - 2\,\frac{x^2-5x+6}{(x-1)^3} = \frac{(2x-5)(x-1)-2(x^2-5x+6)}{(x-1)^3}$$

$$= \frac{\cancel{2x^2}-7x+5-\cancel{2x^2}+10x-12}{(x-1)^3} = \frac{3x-7}{(x-1)^3} = 0 \iff x = 7/3$$

$x = 7/3$ is a critical point. More precisely, $x = \frac{7}{3}$ is a strict global minimizer.