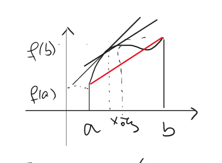
Theorems about derivatives:



Theorem (Lagrange)

Let $a,b \in \mathbb{R}, a<b$. Let $f:[a,b] \to \mathbb{R}$ be a continuous function. Assume that f is differentiable in (a,b). Then $\exists\, \xi \in (a,b)$: $f(b)-f(a) = f'(\xi)(b-a)$.

Corollary (Rolle Theorem)

Let $f:[a,b] \to \mathbb{R}$ be a continuous function. Assume that f is differentiable in (a,b) and $f(a)=f(b) \Rightarrow \exists\, \xi \in (a,b)$: $f'(\xi)=0$.

[$f: X \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable at an interior point $x_0 \in \mathring{X}$ if $\exists \lim\limits_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} =: f'(x_0) \in \mathbb{R}$]

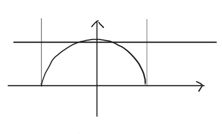
proof: Apply the Lagrange theorem. $\exists\, \xi \in (a,b)$: $f(b)-f(a) = f'(\xi)(b-a)$.

if $f(a)=f(b) \Rightarrow f'(\xi)(b-a)=0 \Rightarrow f'(\xi)=0$.

Remark: . The Rolle theorem gives us a criterium to find critical points of a function.

- The differentiability is not required to hold in the close interval.

$f(x) = \sqrt{1-x^2}$, $\forall x \in [-1,1] = D(f)$



$f'(x) \overset{?}{=}$

$f(x) = (1-x^2)^{1/2} \Rightarrow f'(x) = \underset{\substack{\uparrow \\ \text{chain rule}}}{\frac{1}{2}} (1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}$

$\lim\limits_{x \to 1} f'(x) = \lim\limits_{x \to 1} \frac{-x}{\sqrt{1-x^2}} = \frac{-1}{0^+} = -\infty$, $\lim\limits_{x \to -1} f'(x) = +\infty$

f is not differentiable at ± 1; In any case the conclusion of the Rolle th is still true. [$f'(0)=0$]

De l'Hôpital theorem

Let $f,g: X \subseteq \mathbb{R} \to \mathbb{R}$, let $x_0 \in \mathcal{A}(X)$. Assume that $\exists$ a neighbourhood $I \subseteq \mathbb{R}$ of x_0 such that f and g are differentiable in I. Assume that

$\exists \lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)} \in \overline{\mathbb{R}}$; Then $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)}$

Examples:

$\lim\limits_{x \to 0} \frac{7x - \sin x}{x^2 + \sin(3x)} \overset{?}{=}$

$f(x) = 7x - \sin x$; $g(x) = x^2 + \sin(3x) \Rightarrow f'(x) = 7 - \cos x$; $g'(x) = 2x + 3\cos(3x)$

$x \mapsto 3x \mapsto \sin(3x)$

$\lim\limits_{x \to 0} \frac{f'(x)}{g'(x)} = \lim\limits_{x \to 0} \frac{7 - \cos x}{2x + 3\cos(3x)} = \frac{6}{2 \cdot 0 + 3 \cdot \cos 0} = \frac{6}{3} = 2$

$\Rightarrow \lim\limits_{x \to 0} \frac{7x - \sin x}{x^2 + \sin 3x} = 2$.

- $\lim\limits_{x \to 0^+} (1+2x)^{\frac{1}{\sin x}} = 1^{1/0^+} = 1^{+\infty}$ we were unlucky ☹

$\lim\limits_{x \to 0^+} (1+2x)^{\frac{1}{\sin x}} = \lim\limits_{x \to 0^+} e^{\log\left((1+2x)^{\frac{1}{\sin x}}\right)} = \lim\limits_{x \to 0^+} e^{\frac{\log(1+2x)}{\sin x}}$

$\log(a^b) = b \log a \qquad \forall a>0, \forall b \in \mathbb{R}$

$f(x) = \log(1+2x)$, $g(x) = \sin x$; $f'(x) = \underset{\substack{\uparrow \\ \text{chain rule}}}{\frac{1}{1+2x}} \cdot 2$; $g'(x) = \cos x$

$x \mapsto \underbrace{1+2x}_{=:y} \mapsto \log(1+2x) = \log y$; $(\log y)' = \frac{1}{y}$

$\lim\limits_{x \to 0^+} \frac{f'(x)}{g'(x)} = \lim\limits_{x \to 0^+} \frac{2}{1+2x} \cdot \frac{1}{\cos x} = 2 \Rightarrow \lim\limits_{x \to 0^+} (1+2x)^{\frac{1}{\sin x}} = \lim\limits_{x \to 0^+} e^{\frac{\log(1+2x)}{\sin x} \nearrow 2} = e^2$.

The relation between derivatives and monotonicity

Theorem Let $I \subseteq \mathbb{R}$ be an interval. Let $f: I \to \mathbb{R}$ be a continuous function. If f is differentiable in $\mathring{I}$, then the following is true.

(i) f is non-decreasing $\Leftrightarrow f'(x) \geq 0 \quad \forall x \in \mathring{I}$.

(ii) f is non-increasing $\Leftrightarrow f'(x) \leq 0 \quad \forall x \in \mathring{I}$.

proof: (i). f is non-decreasing $\Rightarrow \frac{f(x+h)-f(x)}{h} \geq 0 \quad \forall x \in \mathring{I}, \forall h \in (-\delta, \delta)$, provided δ is small enough. $\Rightarrow f'(x) \geq 0$.

- $f'(x) \geq 0 \ \forall x \in \mathring{I} \Rightarrow f$ is non-decreasing. Take $a,b \in I, b>a \Rightarrow f$ is diff. in $(a,b) \subseteq I \Rightarrow$ by the Lagrange theorem, $\exists\, \xi \in (a,b)$: $f(b)-f(a) = \underbrace{f'(\xi)}_{\geq 0}\underbrace{(b-a)}_{\geq 0} \geq 0$

Corollary: Let $I \subseteq \mathbb{R}$ be an interval. Assume that $f: I \to \mathbb{R}$ is continuous and differentiable in $\mathring{I}$. Then the following holds:

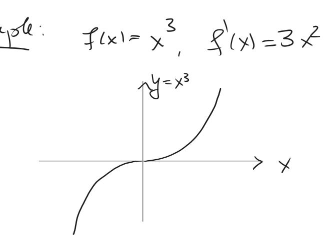
(i) If $f'(x) > 0, \forall x \in \mathring{I} \Rightarrow f$ is strictly increasing in I.

(ii) If $f'(x) < 0, \forall x \in \mathring{I} \Rightarrow f$ is strictly decreasing in I.

proof: Use once again the Lagrange theorem and the fact that $f'>0$ (or $f'<0$).

The opposite implication of Corollary is not necessarily true

Example: $f(x) = x^3$, $f'(x) = 3x^2$. f is strictly increasing on $\mathbb{R}$, but $f'(0)=0$.



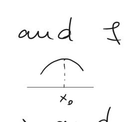
$f'(x_0) \neq 0 \not\Rightarrow x_0$ is either a local min or a local max.

$\Leftarrow$

Definition: Let $f: X \subset \mathbb{R} \to \mathbb{R}$, let $x_0 \in \mathring{X}$. If f is differentiable at x_0 and $f'(x_0)=0$ but x_0 is neither a local minimizer nor a local maximizer, we say that x_0 is a saddle point.

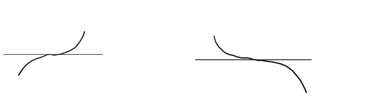
Remark: Let $f: X \subset \mathbb{R} \to \mathbb{R}$ be differentiable at $x_0 \in \mathring{X}$ with $f'(x_0)=0$.

(i) If $\exists\, \delta>0$: $f'(x)>0$ in $(x_0-\delta, x_0)$ and $f'(x)<0$ in $(x_0, x_0+\delta)$, then x_0 is a local maximiser.



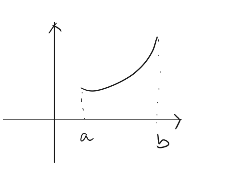
(ii) If $\exists\, \delta>0$: $f'(x)<0$ in $(x_0-\delta, x_0)$ and $f'(x)>0$ in $(x_0, x_0+\delta)$, then x_0 is a local minimizer.

(iii) If $\exists\, \delta>0$: $f'(x)>0$ in $(x_0-\delta, x_0+\delta) \setminus \{x_0\}$ or $f'(x)<0$ in $(x_0-\delta, x_0+\delta)$, then x_0 is a saddle point.

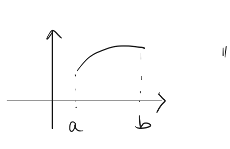


Convexity (or concavity) of a function:

Roughly:



"A function is convex if its graph is like a spoon".



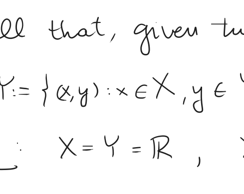
"A function is concave if it is like a reversed spoon".

We recall that, given two sets X and Y, the cartesian product is the set

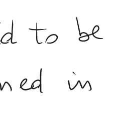
$X \times Y := \{(x,y) : x \in X, y \in Y\}$. [$(x,y) \neq (y,x)$ if $X=Y$]

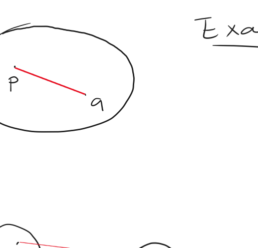
Example: $X = Y = \mathbb{R}$, $X \times Y =: \mathbb{R}^2 = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$ (the cartesian plane).

Definition A set $\Omega \subseteq \mathbb{R}^2$ is said to be convex if $\forall p,q \in \Omega$, the segment connecting p and q is contained in Ω.



Examples: balls are convex, squares are convex triangles are convex

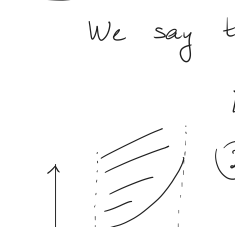




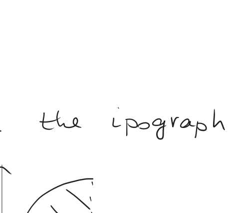
the segment joining p and q is not entirely contained in Ω.

Definition: ① Let $I \subseteq \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$ be a function. We say that f is convex on I if the epigraph

$\{(x,y) \in I \times \mathbb{R} : y > f(x)\}$ is convex.



② We say that f is concave of I if the ipograph $\{(x,y) \in I \times \mathbb{R} : y < f(x)\}$ is convex.



Example: the function $x \mapsto \sin x$ is neither concave nor convex on $\mathbb{R}$

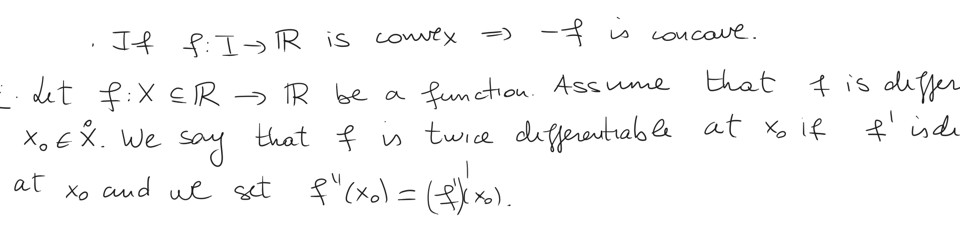
- If f is both concave and convex on $I \Rightarrow f(x) = ax+b$, for some $a,b \in \mathbb{R}$.

- The function $x \mapsto x^2$ is convex.

- If $f: I \to \mathbb{R}$ is convex $\Rightarrow -f$ is concave.

Def: Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Assume that f is differentiable at $x_0 \in \mathring{X}$. We say that f is twice differentiable at x_0 if f' is differentiable at x_0 and we set $f''(x_0) = (f')'(x_0)$.

- If f is twice differentiable at any $x_0 \in \mathring{X}$, we say that f is twice differentiable in $\mathring{X}$.



Theorem Let $f: I \to \mathbb{R}$ be twice differentiable in $\mathring{I}$ (I = interval). Then

(i) f is convex $\Leftrightarrow f''(x) \geq 0 \quad \forall x \in I$

(ii) f is concave $\Leftrightarrow f''(x) \leq 0 \quad \forall x \in I$.