Thm (the derivative of a composition)

Let $X, Y \subseteq \mathbb{R}$ with $\mathring{X} \neq \phi$, $\mathring{Y} \neq \phi$. Let $f: X \to \mathbb{R}$ and $g: Y \to \mathbb{R}$ be function. Let $x_0 \in \mathring{X}$ and assume that $y_0 := f(x_0) \in \mathring{Y}$. If f is differentiable at x_0 and g is differentiable at y_0, then $g \circ f$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0). \quad \text{[Chain rule]}.$$

$$\left[\exists \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} =: f'(x_0) \in \mathbb{R}\right] \qquad [(fg)' = f'g + fg']$$

Examples: $(\tan x)' = \left(\dfrac{\sin x}{\cos x}\right)' = (\sin x \cdot (\cos x)^{-1})' = \dfrac{(\sin x)'}{\cos x} + \sin x\, ((\cos x)^{-1})'$

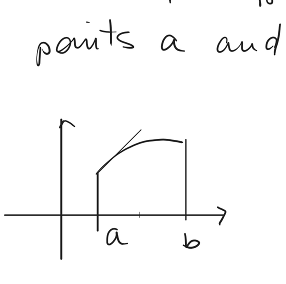
$$= \frac{\cos x}{\cos x} + \sin x \frac{-1}{(\cos x)^2}(-\sin x) = 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x.$$

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)'}{\sin x} + \cos x \left(\frac{1}{\sin x}\right)' = -\frac{\sin x}{\sin x} + \cos x \frac{-1}{\sin^2 x}\cos x = -1 - \cot^2 x$$

$$f(x) = \frac{p(x)}{q(x)} \Rightarrow f'(x) = \frac{p'(x)}{q(x)} + p(x)\left(\frac{1}{q(x)}\right)' = \frac{p'(x)}{q(x)} - \frac{p(x)}{q(x)^2}q'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q^2(x)}$$

Differentiability from the right and from the left:

The typical example is a function $f: [a,b] \to \mathbb{R}$. In this case we already have a notion of differentiability at points $x_0 \in (a,b)$. We want to extend this notion to the points a and b.



Let $X \subseteq \mathbb{R}$ be a set. Assume that $x_0 \in X$ and $\exists\, \delta > 0$: $[x_0, x_0+\delta) \subseteq X$.

$[X = [a,b],\ x_0 = a]$. Let $f: X \to \mathbb{R}$.

Def: We say that f is differentiable from the right at x_0, if

$$\exists \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} =: f'_+(x_0) \in \mathbb{R}$$

Assume that $x_0 \in X$ and $\exists\, \delta > 0$: $(x_0-\delta, x_0] \subseteq X$. Let $f: X \to \mathbb{R}$.

Def: We say that f is differentiable from the left at x_0 if

$$\exists \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} =: f'_-(x_0) \in \mathbb{R}.$$

$[X = [a,b],\ x_0 = b]$

Remark: If $x_0 \in \mathring{X}$, f is differentiable at x_0 $\Leftrightarrow$ f is differentiable both from the right and from the left at x_0 and $f'_-(x_0) = f'_+(x_0) = f'(x_0)$.

Classification of (some) non-differentiability points:

Let $f: X \to \mathbb{R}$ be a function, let $x_0 \in \mathring{X}$. (x_0 is an interior point of $X \Leftrightarrow \exists\, \delta > 0$: $(x_0-\delta, x_0+\delta) \subseteq X$).

Def:

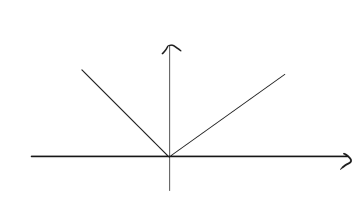
(i) We say that x_0 is an angular point if $\exists f'_-(x_0) \in \mathbb{R}$, $\exists f'_+(x_0) \in \mathbb{R}$ and $f'_-(x_0) \neq f'_+(x_0)$.

(ii) We say that x_0 is a cusp if either $f'_-(x_0) = +\infty$ and $f'_+(x_0) = -\infty$ or $f'_-(x_0) = -\infty$ and $f'_+(x_0) = +\infty$.

(iii) We say that x_0 is a vertical tangent point if $f'_+(x_0) = f'_-(x_0) = +\infty$ or $f'_-(x_0) = f'_+(x_0) = -\infty$.

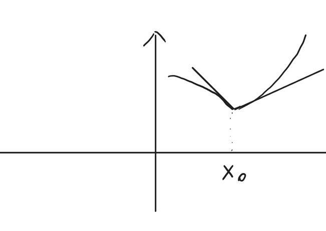
Examples:

(i) $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$



$$f'_+(0) = \lim_{h\to 0^+} \frac{|h|-|0|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1;\quad f'_-(0) = \lim_{h\to 0^-} \frac{|h|-|0|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

$-1 = f'_-(0) \neq f'_+(0) = 1 \Rightarrow x = 0$ is an angular point.



$|x|$ is continuous in $\mathbb{R}$ but not differentiable at $x = 0$.

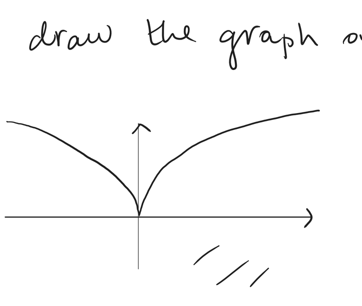
(ii) $f(x) = \sqrt{|x|}$ $\qquad D(f) = \mathbb{R}$; $f(-x) = f(x)$, f is even.

We draw the graph on $[0, +\infty)$

$f(x) = 0 \Leftrightarrow x = 0$

$f(x) > 0\ \forall x \in (0, \infty)$

$\lim_{x\to+\infty} f(x) = +\infty$



$$x > 0;\quad f'(x) = (\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$$

$$f'_+(0) = \lim_{x\to 0^+} f'(x) = \frac{1}{0^+} = +\infty$$

$$f'_-(0) = -\infty$$

In particular $x = 0$ is a cusp.

An equivalent way to see that is the following:

$$\cdot \lim_{h\to 0^+} \frac{\sqrt{|h|}-0}{h} = \lim_{h\to 0} \frac{\sqrt{h}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = +\infty$$

$$\cdot \lim_{h\to 0^-} \frac{\sqrt{|h|}-0}{h} = \lim_{h\to 0^-} \frac{\sqrt{-h}}{-(-h)} = \lim_{h\to 0^-} \frac{1}{-\sqrt{(-h)}} = -\infty$$

(iii) $f(x) = \sqrt[3]{x}$, $x = 0$

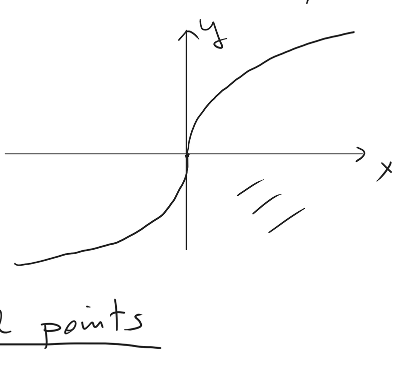
$$\cdot f'_+(0) = \lim_{h\to 0^+} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0^+} \frac{1}{h^{2/3}} = \frac{1}{0^+} = +\infty$$

$$\cdot f'_-(0) = \lim_{h\to 0^-} \frac{\sqrt[3]{+h}}{h} = \lim_{h\to 0^-} \frac{1}{h^{2/3}} = \lim_{h\to 0^-} \frac{1}{\sqrt[3]{h^2}} = \frac{1}{0^+} = +\infty$$

Graph: $D(f) = \mathbb{R}$; f is odd; $f(x) = 0 \Leftrightarrow x = 0$; $f(x) > 0\ \forall x > 0$; $\lim_{x\to+\infty} f(x) = +\infty$

$$f'(x) = (x^{\frac{1}{3}})' = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3}$$

$$\lim_{x\to 0^+} f'(x) = +\infty$$



Critical points

Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Assume that f is differentiable in $\mathring{X} \neq \phi$.

Def: $x_0 \in \mathring{X}$ is a critical point of f if $f'(x_0) = 0$.

Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let $x_0 \in X$.

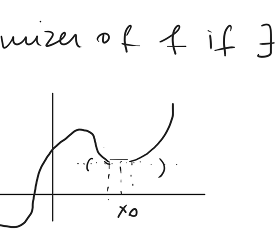
Def: (i) x_0 is a local minimizer of f if $\exists\, \delta > 0$: $\forall x \in (x_0-\delta, x_0+\delta) \cap X$, $f(x) \geq f(x_0)$.

(ii) x_0 is a strict local minimizer of f if $\exists\, \delta > 0$: $\forall x \in (x_0-\delta, x_0+\delta) \cap X$, $f(x) > f(x_0)$.

(iii) x_0 is a local maximizer of f if $\exists\, \delta > 0$: $\forall x \in (x_0-\delta, x_0+\delta) \cap X$, $f(x) \leq f(x_0)$.

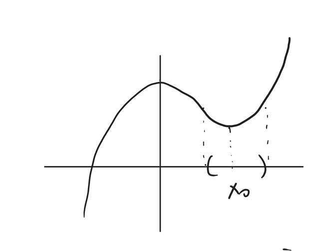
(iv) x_0 is a strict local maximizer of f if $\exists\, \delta > 0$: $\forall x \in (x_0-\delta, x_0+\delta) \cap X$, $f(x) < f(x_0)$.

Examples: (i) local minimizer



x_0 is a local minimizer.

(ii) strict local minimizer



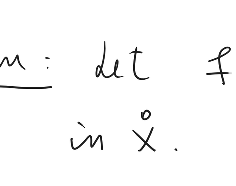
Remark: if x_0 is a strict local minimizer $\Rightarrow$ x_0 is a local minimizer

the same for maximizers.

Remark: if $x_0 \in \mathring{X}$ is a minimizer (or global minimizer) $\Rightarrow$ x_0 is a local minimizer

$\not\Leftarrow$

$f(x) = x^2$



$x = 0$ is a minimizer, $f(0) = \min_{\mathbb{R}} f = \inf_{\mathbb{R}} f$.

Theorem: Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Assume that f is differentiable in $\mathring{X}$. If $x_0 \in \mathring{X}$ is a local minimizer or a local maximiser, then $f'(x_0) = 0$.

proof: $\exists\, \delta > 0$: $(x_0-\delta, x_0+\delta) \subseteq X$, since $x_0 \in \mathring{X}$. Let $h \in (-\delta, \delta) \setminus \{0\}$.

$$\frac{f(x_0+h)-f(x_0)}{h} \begin{cases} \geq 0 & \text{if } h > 0 \quad (x_0 = \text{local minimizer}) \\ \leq 0 & \text{if } h < 0 \end{cases}$$

$\Rightarrow f'_-(x_0) \leq 0$ and $f'_+(x_0) \geq 0$; f is differentiale, so $f'_-(x_0) = f'_+(x_0) = f'(x_0)$

$\Rightarrow f'_-(x_0) = f'_+(x_0) = 0 = f'(x_0)$.