

Polynomials and rational functions.

Def: a function $p: \mathbb{R} \rightarrow \mathbb{R}$ is said to be a polynomial if

$\exists m \in \mathbb{N}$ and $a_0, \dots, a_m \in \mathbb{R}$, with $a_m \neq 0$ if $m > 0$, such that

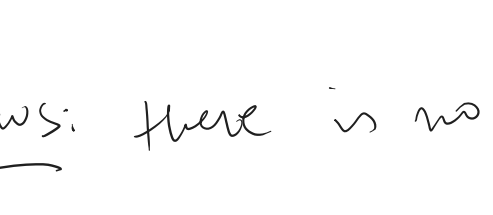
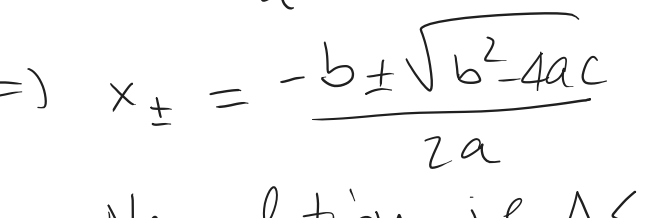
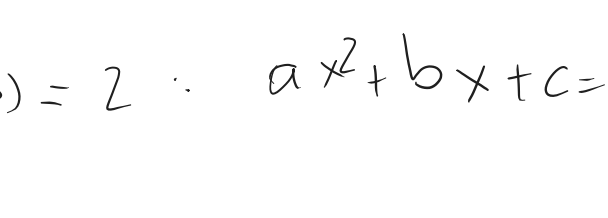
$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_mx^m.$$

If $m = 0$, $p(x) \equiv \text{constant}$; If $m > 0$ $p \neq \text{const.}$

In these notations, m is known as the degree of the polynomial.

So, for example, polynomials of degree 0 are constant functions.

polynomials of degree 1 are straight lines, polynomials of degree 2 are parabolas.



We want to solve a polynomial equation of the form $p(x) = 0$. If $\deg(p) \leq 2 \Rightarrow$ we can solve it.

$\deg(p) = 1: ax + b = 0, a \neq 0 \Rightarrow x = -\frac{b}{a}$

$\deg(p) = 2: ax^2 + bx + c = 0, a \neq 0 \Rightarrow x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, if $\Delta = b^2 - 4ac \geq 0$

No solution if $\Delta < 0$

Bad news: there is no general formula.

Factorization theorem: let $p: \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial of degree at least 3. Then p can be expressed as the product of polynomials of degree 1 and 2.

Given a polynomial p , $\exists \alpha_1, \dots, \alpha_k \in \mathbb{R}$, $\delta_1, \dots, \delta_k \in \mathbb{N}$ of a constant $c \in \mathbb{R}$, and polynomials $q_1, \dots, q_k$ of degree 2:

$$p(x) = c(x - \alpha_1)^{\delta_1} \dots (x - \alpha_k)^{\delta_k} q_1(x) \dots q_k(x).$$

Moreover, we can always assume that $\Delta(q_i) < 0, \forall i = 1, \dots, k$.

In fact, if $\exists i \in \{1, \dots, k\}: \Delta(q_i) \geq 0 \Rightarrow q_i$ can be written as a product of 2 polynomials of degree 1.

If $\Delta(q_i) > 0 \Rightarrow$ equation $q_i = 0$ has exactly 2 solutions $x_{\pm}$.

$$\Rightarrow q_i(x) = c(x - x_-)(x - x_+).$$

If $\Delta(q_i) = 0 \Rightarrow$ equation $q_i = 0$ has 1 solution x_0 and $q_i = c(x - x_0)^2$

In conclusion, the solutions to equation $p(x) = 0$ are $\alpha_1, \dots, \alpha_k$.

Example: $p(x) = x^3 - 2x^2 - x + 2 = x^2(x-2) - (x-2) = (x-2)(x^2-1) = (x-2)(x-1)(x+1)$

$$\boxed{a^2 - b^2 = (a+b)(a-b)}$$

the solutions to equation $p(x) = 0$

$$\text{are } x = 2 \vee x = 1 \vee x = -1$$

$$p(x) = x^3 - x^2 - x + 1 = x^2(x-1) - (x-1) = (x-1)(x^2-1) = (x-1)(x-1)(x+1)$$

$$p(x) = (x-1)^2(x+1)$$

the solutions to equation $p(x) = 0$ are $x = 1 \vee x = -1$.

$$p(x) = x - x^3 = x(1 - x^2) = (1+x)(1-x)x$$

the solutions to $p(x) = 0$ are $x = 0 \vee x = 1 \vee x = -1$.

Polynomial inequalities: Given a polynomial $p: \mathbb{R} \rightarrow \mathbb{R}$ of degree at least 1, we are interested to understand for which values of $x \in \mathbb{R}$, we have $p(x) > 0$, or $p(x) < 0$, or $p(x) \geq 0$, or $p(x) \leq 0$.

If $\deg(p) \leq 1$ we already know how to do.

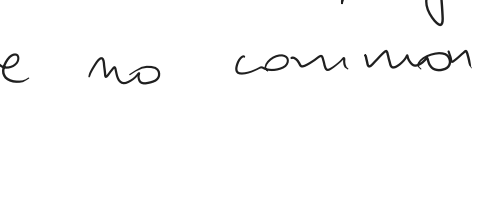
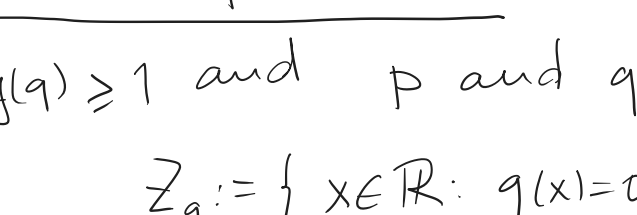
$\deg(p) = 1, ax + b > 0, a \neq 0$. Up to changing the sign of a , we can always assume that $a > 0$. Hence the inequality is true if $x > -b/a$.

$\deg(p) = 2, ax^2 + bx + c > 0, a \neq 0$. Up to changing the sign of a and the inequality, we can assume that $a > 0$.

$\Delta = 0$, then the inequality is true if $x \neq -\frac{b}{2a}$

$\Delta > 0$, then the inequality is true if $x < x_- \vee x > x_+$

$\Delta < 0$, then the inequality is always true.



$\deg(p) \geq 3$.

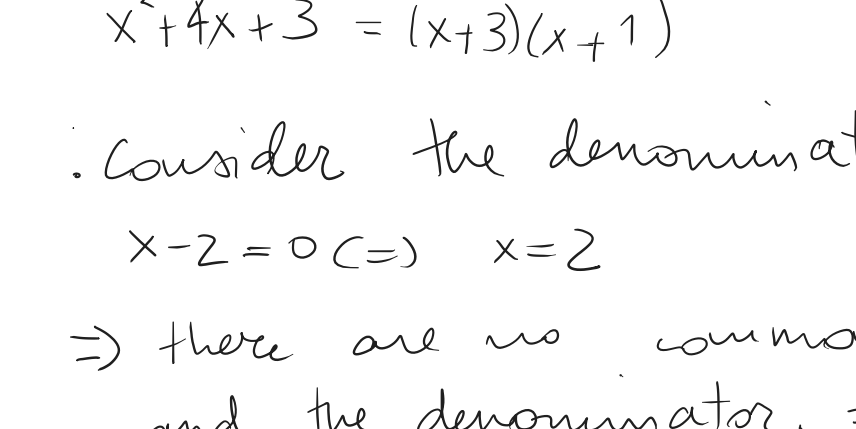
① we factorize the polynomial.

② we solve equation $p(x) = 0$.

③ we calculate the sign of $p(x)$ when x is not a root (or a zero) of the polynomial.

Examples: $p(x) = x - x^3 = x(1 - x^2) = x(1 - x)(1 + x)$

the roots of p are $x = 0, x = 1, x = -1$ [the roots of p are the solutions to $p(x) = 0$]



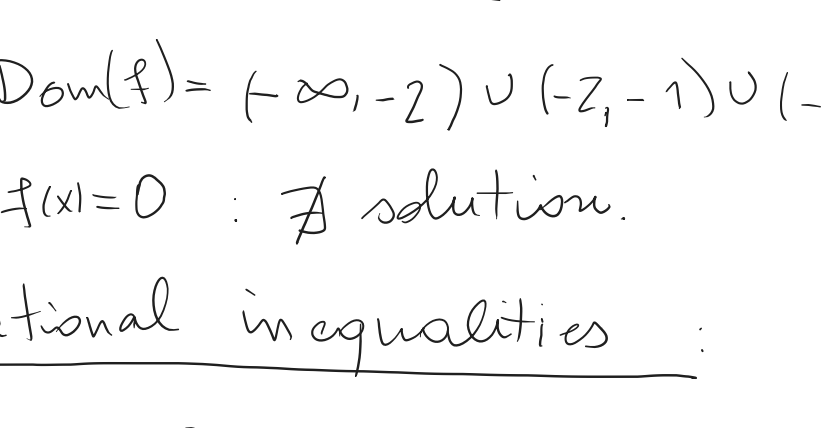
x
 $1-x$
 $1+x$

$$x - x^3 > 0 \text{ for } x \in (-\infty, -1) \cup (0, 1)$$

$$x - x^3 > 0 \text{ for } x \in (-\infty, -1] \cup [0, 1]$$

$$x - x^3 < 0 \text{ for } x \in (-1, 0) \cup (1, \infty)$$

$$p(x) = x^3 - 2x^2 - x + 2 = (x-1)(x+1)(x-2)$$



$x-1$
 $x+1$
 $x-2$

$$p(x) > 0 \text{ for } x \in (-1, 1) \cup (2, \infty)$$

Rational functions: let $p, q: \mathbb{R} \rightarrow \mathbb{R}$ be polynomials, assume that $\deg(q) \geq 1$ and p and q have no common factors. Set

$$Z_q = \{x \in \mathbb{R}: q(x) = 0\}$$

and $f: \mathbb{R} \setminus Z_q \rightarrow \mathbb{R}$

$$f(x) = \frac{p(x)}{q(x)}.$$

A function of this form is known as a rational function.

we are interested in rational equations, that is equations of the form $f(x) = 0$, where f is a rational function, and rational inequalities, that is inequalities of the form $f(x) > 0$ or $f(x) < 0$ or $f(x) \geq 0$ or $f(x) \leq 0$, where f is a rational function.

Examples: $f(x) = \frac{x^2 + 4x + 3}{x-2}$

we want to solve equation $f(x) = 0$.

Steps:

① Can we simplify the fraction? No

$$x^2 + 4x + 3 = (x+3)(x+1) \quad x_{\pm} = \frac{-4 \pm \sqrt{16-43}}{2} = \frac{-4 \pm 1}{2} = -3, -1$$

Consider the denominator:

$$x-2 = 0 \Rightarrow x = 2$$

$\Rightarrow$ there are no common factors between the numerator and the denominator. $\Rightarrow$ we cannot simplify.

② $\text{Dom}(f) = \{x \in \mathbb{R}: x-2 \neq 0\} = \mathbb{R} \setminus \{2\}$

③ the solutions to $f(x) = 0$ are exactly the solutions to numerator = 0, that is, in this case, $x = -1$ and $x = -3$.

Example: $f(x) = \frac{x^2 + 3x + 4}{x+1} = \frac{(x+1)(x+3)}{x+1} = x+3$

$$f(x) = 0 \Rightarrow x+3 = 0 \Rightarrow x = -3$$

Example: $f(x) = \frac{x-1}{(x^2-1)(x+2)} = \frac{x-1}{(x-1)(x+1)(x+2)} = \frac{1}{(x+1)(x+2)}, \forall x \neq 1$.

$$\text{Dom}(f) = (-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, \infty)$$

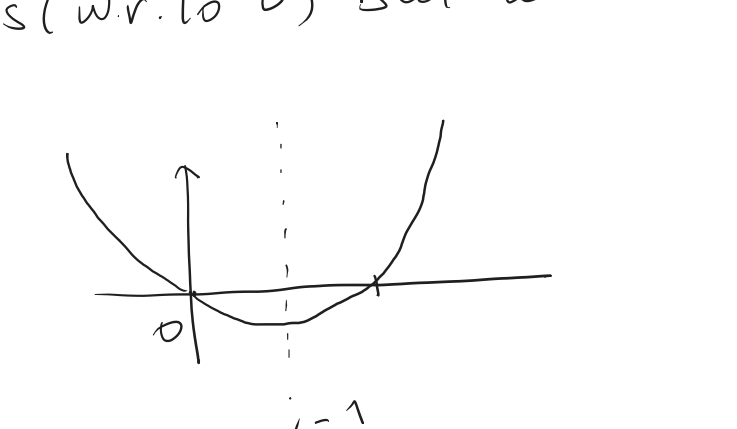
$$f(x) = 0: \text{no solution.}$$

Rational inequalities:

$$f(x) = \frac{x^2 + 4x + 3}{x-2} > 0$$

$$f(x) = \frac{(x+3)(x+1)}{x-2} > 0$$

$$\text{for } x \in (-3, -1) \cup (2, \infty)$$



$x-2$
 $x+3$
 $x+1$

$$f(x) \geq 0 \text{ for } x \in [-3, -1] \cup (2, +\infty)$$

$$f(x) = \frac{x-1}{(x^2-1)(x+2)} > 0$$

$$f(x) = \frac{x-1}{(x-1)(x+1)(x+2)} = \frac{1}{(x+1)(x+2)} \quad \forall x \neq 1$$

$$\frac{1}{(x+1)(x+2)} > 0, x \neq 1$$



$x+1$
 $x+2$

the solution $x \in (-\infty, -2) \cup (-1, \infty) \setminus \{1\}$.

Qualitative properties of functions:

Monotonicity: let $X \subseteq \mathbb{R}$ be a set, let $f: X \rightarrow \mathbb{R}$ be a function. Then we say that

f is non-decreasing if $\forall x_1, x_2 \in X, x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

f is strictly increasing if $\forall x_1, x_2 \in X, x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

f is non-increasing if $\forall x_1, x_2 \in X, x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

f is strictly decreasing if $\forall x_1, x_2 \in X, x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.



non-dec

str. inc

non-inc

str. dec

Definition: A set $X \subseteq \mathbb{R}$ is symmetric with respect to 0 if $x \in X \Rightarrow -x \in X$, where $-X = \{-x: x \in X\}$

For example, $X = [0, 1] \Rightarrow -X = [-1, 0] \Rightarrow X$ is not symmetric with respect to 0.

$X = [-1, 1]$ is symmetric with respect to 0.

Definition: let $X \subseteq \mathbb{R}$ be symmetric with respect to 0. let $f: X \rightarrow \mathbb{R}$ be a function. Then we say that

f is even if $f(x) = f(-x) \quad \forall x \in X$.

f is odd if $f(x) = -f(-x) \quad \forall x \in X$.



f even $\Leftrightarrow$ its graph is symmetric w.r to the y-axis



f odd $\Leftrightarrow$ its graph is symmetric w.r to the origin.

Examples: $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$ is even.

$f(x) = x^3$ is odd

$f(x) = x - x^3$ is odd [$f(x) = -x - (-x)^3 = -x - (-1)^3 x^3 = -x + x^3 = -(x - x^3) = -f(x)$]

Remark: $\exists$ functions which are defined on symmetric domains (w.r to 0) but which are neither odd nor even.

example: $f(x) = x(x-2) = x^2 - 2x$

