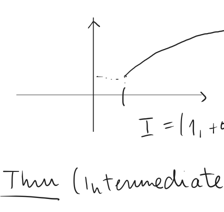
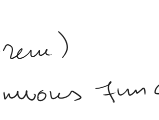
Theorem (Weierstraß) Let $X \subseteq \mathbb{R}$ be a closed and bounded set. Let $f: X \to \mathbb{R}$ be a continuous function. Then f has a minimum and a maximum in X.


$X = [a,b]$

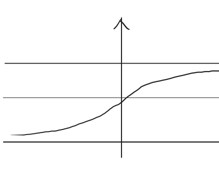
$f(x) = 2x+1, \ \forall x \in [0,1]$



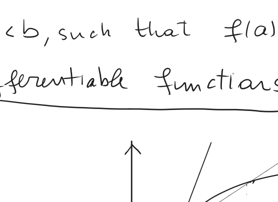
$\max_{x\in[0,1]} f(x) = f(1) = 3 = 2\cdot 1 + 1$

$\min_{x\in[0,1]} f(x) = f(0) = 2\cdot 0 + 1 = 1$

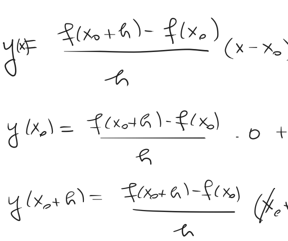
Thm Let $I \subseteq \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$ be continuous. Then $f(I)$ is either a point or an interval.


$I = (1, +\infty)$

Thm (Intermediate values theorem)
Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function, with I = an interval. Let $y \in \mathbb{R}$ be such that $\inf_{x\in I} f(x) \le y \le \sup_{x \in I} f(x)$. Then $\exists x_0 \in I: \ y = f(x_0)$.



$X := [0,1] \cup [2,3]$



$f: X \to \mathbb{R}$
$x \mapsto 2x+1$

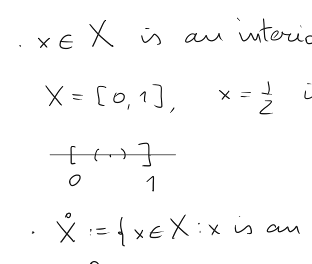
$\inf_X f = 1 = f(0)$

$\sup_X f = f(3) = 7$

$y = 4. \ \nexists x \in X: f(x) = 4$

Thm (Zero theorem)
Let $I \subseteq \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$ be continuous. Assume that $\exists a, b \in I$, $a<b$, such that $f(a)\cdot f(b) < 0$. Then $\exists x_0 \in (a,b): \ f(x_0) = 0$.

Differentiable functions



$y(x) = \frac{f(x_0+h)-f(x_0)}{h}(x-x_0) + f(x_0), \ h>0$

$y(x_0) = \frac{f(x_0+h)-f(x_0)}{h}\cdot 0 + f(x_0) = f(x_0)$

$y(x_0+h) = \frac{f(x_0+h)-f(x_0)}{h}(\cancel{x_0}+h-\cancel{x_0}) + f(x_0)$

$= \frac{f(x_0+h)-f(x_0)}{\cancel{h}}\cdot\cancel{h} + f(x_0)$

$= f(x_0+h) - f(x_0) + f(x_0) = f(x_0+h)$

Let $X \subset \mathbb{R}$ be a set with $\mathring{X} \neq \emptyset$.

Recall: . $x \in X$ is an interior point if $\exists \delta > 0: \ (x_0 - \delta, x_0+\delta) \subseteq X$.

$X = [0,1]$, $x = \frac{1}{2}$ is an int. point of X, 0 is not an int. point of X.

. $\mathring{X} := \{x \in X : x \text{ is an interior point of } X\}$.

Example: . $\mathring{[0,1]} = (0,1)$

. $\mathring{\{1,2\}} = \emptyset$

Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Assume that $x_0 \in \mathring{X} \Rightarrow \exists \delta > 0$: the function $h \in (-\delta, \delta) \setminus \{0\} \mapsto \frac{f(x_0+h)-f(x_0)}{h}$ is well defined. This function is known as the incremental ratio.

Def: . Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Assume that $\mathring{X} \neq \emptyset$ and $x_0 \in \mathring{X}$. We say that f is differentiable at x_0 if $\exists \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} =: f'(x_0)$ and $f'(x_0) \in \mathbb{R}$.

. In particular, if f is differentiable at x_0, $f'(x_0)$ is known as the derivative of f at x_0.

. f is differentiable in $\mathring{X}$ if f is differentiable at x_0, $\forall x_0 \in \mathring{X}$.

Theorem (relation between differentiability and continuity)
Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Assume that $\mathring{X} \neq \emptyset$ and f is differentiable in $\mathring{X}$. Then f is continuous in $\mathring{X}$.

Examples: $f(x) = \begin{cases} 2x & x > 0 \\ 1 & x = 0 \end{cases}$

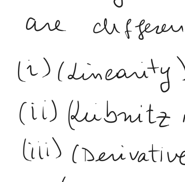
f is not continuous in $[0, +\infty)$. But f is differentiable in $(0, +\infty)$.

. $x>0$, $\lim_{h\to 0} \frac{2(x+h)-2x}{h} = \lim_{h\to 0}\frac{\cancel{2x}+2h-\cancel{2x}}{h} = \lim_{h\to0}\frac{2\cancel{h}}{\cancel{h}} = \lim_{h\to 0} 2 = 2$

$\forall x > 0$, f is differentiable at x and $f'(x) = 2$.

. **Remark:** Let $f(x) := \alpha x + \beta$, with $\alpha, \beta \in \mathbb{R}$, $x \in \mathbb{R}$. $\Rightarrow f$ is differentiable in $\mathbb{R}$ and $f'(x) = \alpha$

Roughly: $f'(x)$ is the slope of the graph at x.



. **Example:** $f(x) = x^2$

$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h} = \lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x$

f is differentiable in $\mathbb{R}$ and $f'(x) = 2x$.

. **Thm** Let $f, g: X \subseteq \mathbb{R} \to \mathbb{R}$ be functions. Assume that $\mathring{X} \neq \emptyset$, $x_0 \in \mathring{X}$. Assume that f and g are differentiable at x_0. Let $\alpha, \beta \in \mathbb{R}$. Then

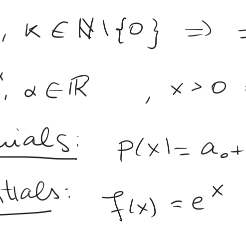
(i) (Linearity) $\alpha f + \beta g$ is differentiable at x_0 and $(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0)$.

(ii) (Leibnitz rule) $f\cdot g$ is differentiable at x_0 and $(f\cdot g)'(x_0) = f'(x_0) g(x_0) + f(x_0) g'(x_0)$.

(iii) (Derivative of the inverse function). If $\exists \delta > 0$: f is invertible in $(x_0-\delta, x_0+\delta)$, then f^{-1} is differentiable at $y_0 := f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

Definition: Let $X \subseteq \mathbb{R}$ be such that $\mathring{X} \neq \emptyset$. Let $f: X \to \mathbb{R}$ be differentiable at x_0. Then the tangent line to the graph of f (or to f) at x_0 is the straight line of equation $y(x) = f'(x_0)(x-x_0) + f(x_0)$



"The tangent line is the straight line which approximates f in a neighbourhood of x_0".

Some derivatives of known functions:

Examples:

1. $f(x) = \alpha x + \beta$, $\alpha, \beta \in \mathbb{R} \Rightarrow f'(x) = \alpha$.

2. $f(x) = x^k$, $k \in \mathbb{N}\setminus\{0\} \Rightarrow f'(x) = k x^{k-1}$ [Leibnitz rule]

3. $f(x) = x^\alpha$, $\alpha \in \mathbb{R}$, $x > 0 \Rightarrow f'(x) = \alpha x^{\alpha - 1}$.

4. Polynomials: $P(x) = a_0 + \dots + a_n x^n \Rightarrow p(x) = a_1 + \dots + n a_n x^{n-1}$

5. exponentials: $f(x) = e^x$

$f'(x) = \lim_{h\to0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to0}\frac{e^x(e^h-1)}{h} = e^x \lim_{h\to0}\frac{e^h-1}{h} = e^x$

6. Logarithms: $f(x) = \log x$. $f = g^{-1}$, $g(x) = e^x$

$f'(x) = \frac{1}{g'(g^{-1}(x))} = \frac{1}{e^{\log x}} = \frac{1}{x}$

7. sin and cos: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$.

$f(x) = \sin x$

$f'(x) = \lim_{h\to0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h} = \lim_{h\to0}\left(\sin x \underbrace{\frac{\cos h-1}{h}}_{\to 0} + \cos x \underbrace{\frac{\sin h}{h}}_{\to 1}\right)$

$= \cos x$

$(\cos x)' = -\sin x$.

Exercises:

Sheet 6,

1,a) $\lim_{x\to\infty}\frac{2x^3-3x^2}{x^2(x^2+1)} = \lim_{x\to+\infty}\frac{x^3(2-3\frac{x^2}{x^3})}{x^2\cdot x^2(1+\frac{1}{x^2})} = \lim_{x\to\infty}\frac{2\overbrace{(-\frac{3}{x})}^{\to 0}}{x(1+\underbrace{\frac{1}{x^2}}_{\to 0})} = \frac{2}{+\infty\cdot 1} = 0^+$

2,a) $f(x) = \log_3\left(\frac{x^2+1}{x^2-1}\right)$. Draw the graph.

I. $D(f) = (-\infty, -1) \cup (1, +\infty)$

$\frac{x^2+1}{x^2-1} > 0 \iff x > 1 \lor x < -1$

$x^2 - 1 = (x-1)(x+1) > 0$

$x^2 + 1 > 0 \ \forall x \in \mathbb{R}$

		-1		1	
$x+1$	$-$		$+$		$+$
$x-1$	$-$		$-$		$+$
	$+$		$-$		$+$

$x+1>0 \iff x>-1$

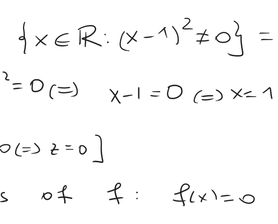
$x-1>0 \iff x>1$

II. Symmetries: f is even.

III. The sign: $f(x) > 0 \iff \log_3\left(\frac{x^2+1}{x^2-1}\right) > 0 \quad \forall x \in D(f)$.

$\log_3 z > 0 \iff 3^{\log_3 z} > 3^0 \iff z > 1$

we need to ask $\frac{x^2+1}{x^2-1} > 1$, $x \in D(f)$, $x>0 \iff x^2+1 > x^2-1$, $x\in D(f), x>0$. This is true $\forall x \in D(f), x>0$.


$x=1$ $\quad x=1$

IV. limits:

$\lim_{x\to+\infty}\log\left(\frac{x^2+1}{x^2-1}\right) = 0^+ \Rightarrow f$ has a horizontal asymptote $(y=0)$

$\lim_{x\to1^+}\log\left(\frac{x^2+1}{x^2-1}\right) = +\infty \Rightarrow f$ has a vertical asymptote $(x=1)$

. By symmetry, f has 2 vertical asymptotes $x=1$ and $x=-1$

2,b) $f(x) = \frac{x^2-5x+6}{(x-1)^2}$

I. $D(f) = \{x \in \mathbb{R} : (x-1)^2 \neq 0\} = \mathbb{R}\setminus\{1\}$

$(x-1)^2 = 0 \iff x-1=0 \iff x=1$

$[z^2 = 0 \iff z=0]$

II. Zeros of f: $f(x) = 0 \iff \frac{x^2-5x+6}{(x-1)^2} = 0 \iff x^2-5x+6 = 0 \iff x_\pm = \frac{5\pm\sqrt{25-4\cdot 6}}{2}$

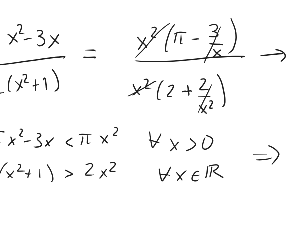
$x_\pm = \frac{5\pm1}{2} = 2/3$

$f(x) = 0 \iff x=2 \lor x=3$.

III. The sign of f: $f(x) > 0 \iff \frac{(x-2)(x-3)}{(x-1)^2} > 0$

$\iff (-\infty, 1) \cup (1,2) \cup (3, \infty)$

		1		2		3	
$x-2$	$-$		$-$		$+$		$+$
$x-3$	$-$		$-$		$-$		$+$
$(x-1)^2$	$+$	0	$+$		$+$		$+$
	$+$		$+$		$-$		$+$


$x=1$

IV. Limits:

. $\lim_{x\to1}\frac{x^2-5x+6}{(x-1)^2} = +\infty$

. $\lim_{x\to\pm\infty}\frac{x^2-5x+6}{(x-1)^2} = 1$

1,b) $\lim_{x\to\infty}\tan\left(\frac{\pi x^2-3x}{2(x^2+1)}\right) = +\infty$

$\pi/2$

$\frac{\pi x^2 - 3x}{2(x^2+1)} = \frac{\cancel{x^2}(\pi - \frac{3}{x})}{\cancel{x^2}(2+\frac{2}{x^2})} \to \frac{\pi}{2}^- \quad x\to\infty$.

$\pi x^2 - 3x < \pi x^2 \quad \forall x > 0$
$2(x^2+1) > 2x^2 \quad \forall x \in \mathbb{R}$
$\Rightarrow \frac{\pi x^2-3x}{2(x^2+1)} < \frac{\pi}{2} \quad \forall x \in (0, +\infty)$