Limits of rational functions:

Let p and q be polynomials of degree n and m respectively ($m,n \in \mathbb{N}\setminus\{0\}$).

$p(x) = a_0 + \dots + a_n x^n$; $q(x) = b_0 + \dots + b_m x^m$. Let $f(x) = \frac{p(x)}{q(x)}$.

$D(f) = \mathbb{R}\setminus Z(q)$. In particular $\mathcal{A}(D(f)) = \overline{\mathbb{R}}$ and $\mathcal{A}(D(f))\setminus D(f) = \{\pm\infty\} \cup Z(q)$.

So, we are interested in computing the limits $\lim_{x\to x_0} f(x)$ $\forall x_0 \in Z(q)$ and for $x_0 = \pm\infty$.

Let us consider

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0} = \lim_{x\to\pm\infty} \frac{x^n\left(a_n + \underbrace{\frac{a_{n-1}}{x}}_{\to 0} + \dots + \underbrace{\frac{a_0}{x^n}}_{\to 0}\right)}{x^m\left(b_m + \underbrace{\frac{b_{m-1}}{x}}_{\to 0} + \dots + \underbrace{\frac{b_0}{x^m}}_{\to 0}\right)}$$

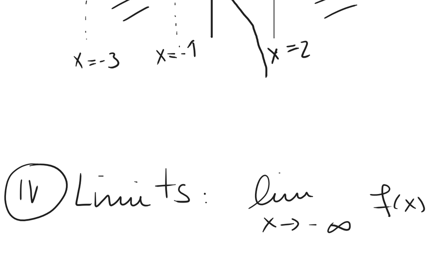
$$= \begin{cases} 0 & \text{if } n < m \\ \frac{a_n}{b_m} \in \mathbb{R}\setminus\{0\} & \text{if } m = n \\ \operatorname{sgn}\left(\frac{a_n}{b_m}\right)\infty & \text{if } n > m \text{ and } n-m \text{ is even.} \\ \pm\operatorname{sgn}\left(\frac{a_n}{b_m}\right)\infty & \text{if } n > m \text{ and } n-m \text{ is odd} \end{cases}$$

Examples: Let $f(x) = \frac{x^2+4x+3}{x-2}$; Draw the graph of f.

(I) $D(f) = \mathbb{R}\setminus\{2\}$.

(II) Determine the zeros of f. $x^2+4x+3=0$ $\Delta = 16-12 = 4$, $x_\pm = \frac{-4\pm 2}{2} = -1/-3$

(III) Determine $f^{-1}((0,+\infty))$, or, in other words, solve inequality $f(x)>0$.



$N>0 \iff x<-3 \vee x>-1$

$D>0 \iff x>2$

	-3	-1	2	
N	+	−	+	+
D	−	−	−	+
	−	+	−	+

	-3	-1	2	
x+3	−	+	+	+
x+1	−	−	+	+
x−2	−	−	−	+
	−	+	−	+

$f(x)>0 \iff x\in(-3,-1)\cup(2,+\infty)$

(IV) Limits: $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+\cancel{3x+4}}{x-\cancel{2}} = \lim_{x\to-\infty} \frac{x^2(1+\frac{3}{x}+\frac{4}{x^2})}{x(1-2/x)} = -\infty$

$\lim_{x\to+\infty} f(x) = +\infty$.

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2+4x+3}{x-2} = \frac{4+8+3}{0^-} = -\infty$

$\lim_{x\to 2^+} f(x) = \frac{15}{0^+} = +\infty$

Question: Does f have any asymptote? $x=2$ is a vertical asymptote.

Theorem (Policemen or comparison theorem).

Let $f,g,h: X\subseteq\mathbb{R}\to\mathbb{R}$ be functions. Assume that $x_0\in\mathcal{A}(X)$, $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) =: \alpha \in \overline{\mathbb{R}}$, and $\exists$ a neighbourhood I of x_0: $f(x)\le h(x)\le g(x)$ $\forall x\in(X\cap I)\setminus\{x_0\}$. Then $\lim_{x\to x_0} h(x) = \alpha$.

Example of application: $\lim_{x\to\infty} \frac{\sin x}{x} = 0$

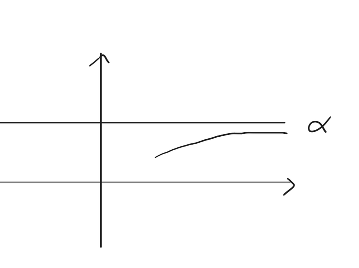
$-1\le\sin x\le 1, \forall x\in\mathbb{R} \Rightarrow -\frac{1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$ $\forall x>0$; $\lim_{x\to+\infty}\left(-\frac{1}{x}\right) = \lim_{x\to+\infty}\frac{1}{x} = 0 \Rightarrow \lim_{x\to\infty}\frac{\sin x}{x} = 0$.

Theorem of the sign

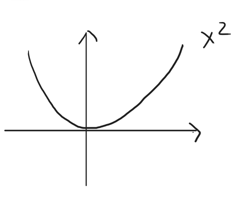
Let $f: X\subseteq\mathbb{R}\to\mathbb{R}$ be a function. Assume that $x_0\in\mathcal{A}(X)$ and $\lim_{x\to x_0} f(x) =: \alpha \in\overline{\mathbb{R}}$.

(i) If $\alpha>0 \Rightarrow \exists$ a neighbourhood $I\subseteq\mathbb{R}$ of x_0: $f(x)>0$, $\forall x\in(I\cap X)\setminus\{x_0\}$.

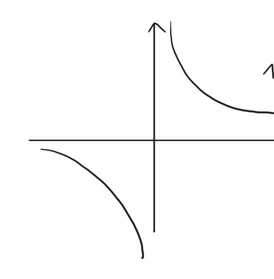
(ii) If $\alpha<0 \Rightarrow \exists$ a neighbourhood $I\subseteq\mathbb{R}$ of x_0: $f(x)<0$, $\forall x\in(I\cap X)\setminus\{x_0\}$.



Continuous functions



continuous



not continuous at $x=0$

Def:
Let $f: X\subseteq\mathbb{R}\to\mathbb{R}$ be a function.

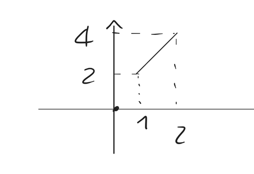
(i) We say that f is continuous at $x_0\in X$ if $\forall\varepsilon>0\ \exists\delta>0: \forall x\in X, |x-x_0|<\delta$, we have $|f(x)-f(x_0)|<\varepsilon$

(ii) We say that f is continuous in X if f is continuous at x_0, $\forall x_0\in X$.

Remark: (1) Assume that $x_0\in X\cap\mathcal{A}(X)$. Then f is continuous at x_0 $\iff \lim_{x\to x_0} f(x) = f(x_0)$.

(2) Assume that $x_0\in X\setminus\mathcal{A}(X)$. Then f is continuous at x_0

Example: $X = \{0\}\cup[1,2]$, $f: X\to\mathbb{R}$, $x\mapsto 2x$



$\mathcal{A}(X) = [1,2]$; 0 is an isolated point of X.

proof (of the Remark, part (2)). $\exists\delta>0: (x_0-\delta, x_0+\delta)\cap X = \{x_0\}$.

if $x\in(x_0-\delta,x_0+\delta)\cap X \Rightarrow x = x_0 \Rightarrow f(x) = f(x_0) \Rightarrow |f(x)-f(x_0)|<\varepsilon, \forall\varepsilon>0$.

Thm: If $f,g: X\subseteq\mathbb{R}\to\mathbb{R}$ are continuous at $x_0\in X \Rightarrow f+g$ and $f\cdot g$ are continuous at x_0.

Thm (Composition of continuous functions)

Let $f: X\subseteq\mathbb{R}\to\mathbb{R}$ and $g: Y\subseteq\mathbb{R}\to\mathbb{R}$ be functions. Assume that f is continuous at $x_0\in X$ and g is continuous at $f(x_0)\in Y$. Then $g\circ f$ is continuous at x_0.

$$X \xrightarrow{f} Y \xrightarrow{g} \mathbb{R}$$
$$x_0 \longmapsto f(x_0) \longmapsto g\circ f(x_0)$$

Classification of discontinuity points:

Let $f: X\subseteq\mathbb{R}\to\mathbb{R}$ be a function and let $x_0\in X\cup\mathcal{A}(X)$. Assume f is not cont at x_0.

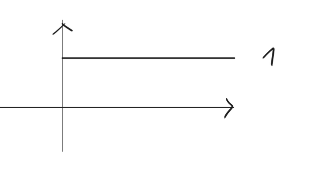
Def: (i) x_0 is a jump point of f if $\exists\alpha := \lim_{x\to x_0^-} f(x)\in\mathbb{R}$, $\exists\beta := \lim_{x\to x_0^+} f(x)\in\mathbb{R}$, but $\alpha\ne\beta$.

(ii) x_0 is a removable discontinuity if $\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0} f(x)\in\mathbb{R}$.

(iii) x_0 is a second-type discontinuity if none of the previous situations occur.

Examples:

(i) $f(x) = \begin{cases} 0 & x<0 \\ 1 & x\ge 0 \end{cases}$



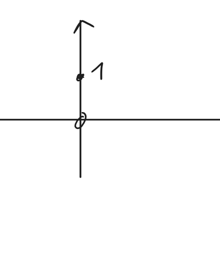
$\lim_{x\to 0^-} f(x) = 0 < 1 = \lim_{x\to 0^+} f(x)$

Definition: If x_0 is a jump point, the amplitude of the jump is

$$\left|\lim_{x\to x_0^+} f(x) - \lim_{x\to x_0^-} f(x)\right|$$

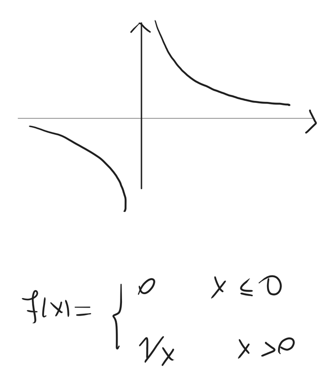
In our previous example, the amplitude of the jump is 1.

(ii) $f(x) = \begin{cases} 0 & x\in\mathbb{R}\setminus\{0\} \\ 1 & x=0 \end{cases}$

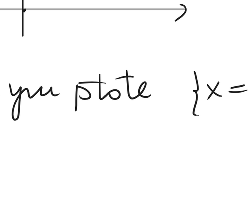


$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0 \ne f(0) = 1$

(iii) $f(x) = \frac{1}{x}$, $D(f) = \mathbb{R}\setminus\{0\}$; $x=0$ is a second type discontinuity.



$f(x) = \begin{cases} 0 & x\le 0 \\ 1/x & x>0 \end{cases}$



$x=0$ is a II-type disc.

Remark: if f has a vertical asymptote $\{x=x_0\}$, then it has a II type discontinuity at x_0.

The relation between continuity and the extrema:

Let $f: X\subseteq\mathbb{R}\to\mathbb{R}$ be a function

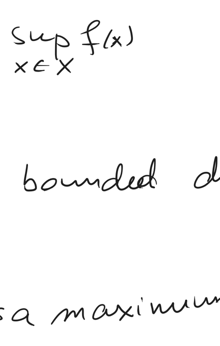
Def: $\sup_{x\in X} f(x) := \sup f(X) = \sup\{f(x): x\in X\}$

$\inf_{x\in X} f(x) := \inf f(X) = \inf\{f(x): x\in X\}$

example:
$f(x) = x^2$, $\inf_{x\in\mathbb{R}} f(x) = \inf f(\mathbb{R}) = \inf[0,+\infty) = 0$

$\sup_{x\in\mathbb{R}} f(x) = +\infty$

$f(\mathbb{R}) = [0,+\infty)$



Theorem (Weierstraß) Let $X\subseteq\mathbb{R}$ be a closed and bounded set. Let $f: X\to\mathbb{R}$ be a continuous function. Then $\exists x_1\in X: f(x_1) = \sup_{x\in X} f(x)$ and $\exists x_2\in X: f(x_2) = \inf_{x\in X} f(x)$.

In other words, a continuous function on a closed and bounded domain has a maximum and a minimum.

Def: Let $f: X\subseteq\mathbb{R}\to\mathbb{R}$ be a function. We say that $x_0\in X$ is a maximum of f if $f(x_0) = \sup_{x\in X} f(x)$ and $x_0\in X$ is a minimum if $f(x_0) = \inf_{x\in X} f(x)$.

In these cases, we write $\inf_{x\in X} f(x) =: \min_{x\in X} f(x)$ and $\sup_{x\in X} f(x) =: \max_{x\in X} f(x)$.