Equations of degree 2:

equations of the form $ax^2+bx+c=0$, with $a,b,c \in \mathbb{R}$, $a \neq 0$.
up to now, we have solved equations of degree 2 with $b=0$.

$ax^2+c=0 \iff ax^2=-c \iff x^2=-\frac{c}{a}$

· If $-\frac{c}{a} \geq 0$, then the equation is solvable and the solutions are $\pm\sqrt{-\frac{c}{a}}$

[in particular if $c=0 \Rightarrow$ the unique solution is $x=0$]

· if $-\frac{c}{a} < 0 \Rightarrow$ there is no solution.

· Example: $x^2-4=0$ has 2 solutions $x_\pm = \pm 2$.

$x^2+4=0 \iff x^2=-4$ $\nexists$ solution in $\mathbb{R}$.

$x^2=0 \iff x=0$

· $ax^2+bx+c=0 \iff x^2+\frac{b}{a}x+\frac{c}{a}=0$

Remark: $(\alpha+\beta)^2=\alpha^2+2\alpha\beta+\beta^2$, $\forall \alpha,\beta \in \mathbb{R}$.

$x^2+2\frac{b}{2a}x+\frac{c}{a}=0 \iff \underbrace{x^2+2\cdot\frac{b}{2a}x+\frac{b^2}{4a^2}}_{(x+\frac{b}{2a})^2}-\frac{b^2}{4a^2}+\frac{c}{a}=0$

$\iff \left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0$

$\iff \left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}$

$\Delta := b^2-4ac$

· if $\Delta=0 \Rightarrow \left(x+\frac{b}{2a}\right)^2=0 \iff x+\frac{b}{2a}=0 \iff x=-\frac{b}{2a}$

$y := x+\frac{b}{2a}$, you solve $y^2=0 \iff y=0 \iff x+\frac{b}{2a}=0 \iff x=-\frac{b}{2a}$

· if $\Delta<0 \Rightarrow \nexists$ solution in $\mathbb{R}$. [because we have $y^2=\frac{\Delta}{4a^2}<0$]

· if $\Delta>0 \Rightarrow$ the solutions are $y_\pm=\pm\frac{\sqrt{\Delta}}{2a}$

$\iff x+\frac{b}{2a}=\frac{\sqrt{\Delta}}{2a} \lor x+\frac{b}{2a}=-\frac{\sqrt{\Delta}}{2a} \iff x=\frac{-b+\sqrt{\Delta}}{2a} \lor x=\frac{-b-\sqrt{\Delta}}{2a}$

$\iff$ the solutions of our equations are $x_\pm=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

An equivalent formulation:

$x_\pm=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-\frac{b}{2}\pm\frac{\sqrt{b^2-4ac}}{2}}{a}=\frac{-\frac{b}{2}\pm\sqrt{\frac{b^2-4ac}{4}}}{a}$

$x_\pm=\frac{-\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2-ac}}{a}=\frac{-\frac{b}{2}\pm\sqrt{\frac{\Delta}{4}}}{a}$

$x^2-2x+3=0$ $\quad x_\pm=\frac{1\pm\sqrt{1-3}}{1}\notin\mathbb{R}$; $\Delta<0 \Rightarrow \nexists$ solution.

$\Delta=4-4\cdot1\cdot3=4-12=-8<0$

· Inequalities: $ax^2+bx+c>0$, $a>0$, $b,c\in\mathbb{R}$

· $\Delta=0$ $\quad ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2$

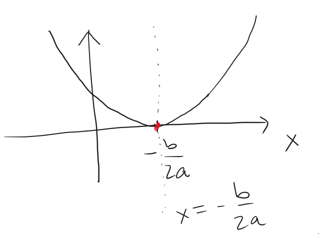
$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left(x^2+2\frac{b}{2a}x+\frac{c}{a}\right)=$

$=a\left(x+2\frac{b}{2a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)$

$=a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b-4ac}{4a^2}\right)=a\left(\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}\right)$

· $\Delta=0$, $a>0 \Rightarrow ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2=ay^2$, $y:=x+\frac{b}{2a}$

$\Rightarrow ax^2+bx+c\geq0$ $\forall x\in\mathbb{R}$, $ax^2+bx+c=0 \iff x=-\frac{b}{2a}$

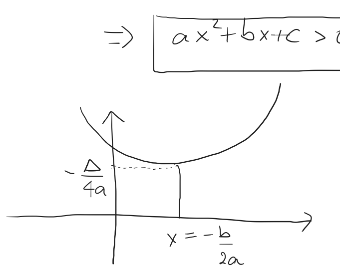


$\boxed{\cdot\ ax^2+bx+c>0 \iff x\neq-\frac{b}{2a} \iff x\in\mathbb{R}\setminus\{-\frac{b}{2a}\}}$

· $ax^2+bx+c<0$ is never satisfied in $\mathbb{R}$

· $\Delta<0$, $a>0 \Rightarrow ax^2+bx+c=\underset{\vee\atop 0}{a}\left(\left(x+\frac{b}{2a}\right)^2-\underbrace{\frac{\Delta}{4a^2}}_{\vee\atop 0}\right)>a\left(x+\frac{b}{2a}\right)^2\geq0$

$\Rightarrow \boxed{ax^2+bx+c>0 \quad \forall x\in\mathbb{R}}$



$f:\mathbb{R}\to\mathbb{R}$

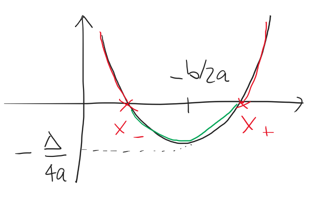
$f(x)=ax^2+bx+c$

$f(x)=a\left(\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}\right)$

$\Rightarrow f\left(-\frac{b}{2a}\right)=a\left(\left(-\cancel{\frac{b}{2a}}+\cancel{\frac{b}{2a}}\right)^2-\frac{\Delta}{4a^2}\right)=-\frac{\Delta}{4a}$

· $\Delta<0, a>0 \Rightarrow ax^2+bx+c<0$ is never satisfied.

· $\Delta>0, a>0 \Rightarrow f(x)=a\left(\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}\right) \Rightarrow f\left(\frac{b}{2a}\right)=-\frac{\Delta}{4a}<0$



in order to find the intersections of the parabola with the x-axis, we need to solve equation $f(x)=0$. In our case the solutions are $ax^2+bx+c=0 \iff x_\pm=\frac{-b\pm\sqrt{\Delta}}{2a}$

· $ax^2+bx+c>0 \iff x>x_+ \lor x<x_-$

· $ax^2+bx+c<0 \iff x_-<x<x_+$

Remark: if $a<0$, and we want to solve $ax^2+bx+c>0$ $(<)$

$\Rightarrow$ we can reduce ourselves to solve $-ax^2-bx-c<0$ $(>)$.

[by multiplying by -1] and changing the inequality.

Exercises:

(2, d) $x(x-1)=2$

$x^2-x-2=0$

$a=1$ $\quad b=-1$ $\quad c=-2$

$\Delta=1-4(1\cdot(-2))$ $\quad \Delta=1+8=9$

$x_1=\frac{1+3}{2}=\frac{4}{2}=2$

$x_2=\frac{1-3}{2}=-1$

(3,a) $x^2+x-2<0$

$x^2+x-2=0$

$a=1$

$b=1$

$c=-2$

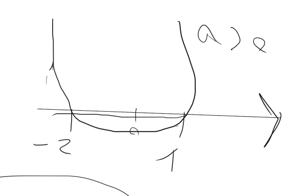
$\Delta=1-(4(1(-2)))=9$

$x_1=\frac{-1+3}{2}=1$

$x_2=\frac{-1-3}{2}=\frac{-4}{2}=-2$

$x_1=1$

$x_2=-2$



$-2<x<1$

· (3,b) $4x^2-4x+1>0$

$4x^2-4x+1=0$

$a=4$ $\quad \Delta=b^2-4ac$

$b=-4$ $\quad \Delta=(-4)^2-4\cdot(4\cdot1)$

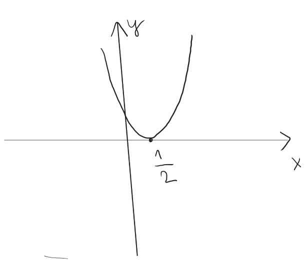
$c=1$ $\quad \Delta=16-4\cdot4=0$

$x_{1,2}\ \frac{-b\pm\sqrt{\Delta}}{2a}$

$x_1\ \frac{-(-4)-0}{2\cdot4}=\frac{\cancel{4}}{\cancel{8}}=\frac{1}{2}$

$x_2\ \frac{-(-4)+0}{2\cdot4}=\frac{1}{2}$

$x_{1,2}=\frac{1}{2}$



$x\in\left(-\infty;\frac{1}{2}\right)\cup\left(\frac{1}{2};+\infty\right)$

(3,d) $2^{-x}+2^{x}+6\geq0$

$2^x>0$ $\forall x\in\mathbb{R}$, $2^{-x}=\frac{1}{2^x}>0$ $\forall x\in\mathbb{R}$, $\Rightarrow$ the inequality is always true.