Exercises: Sheet 9

1,a) $\int \underbrace{\left(4\sin x - \frac{\tan x}{\cos x}\right)}_{f(x)} dx = 4\int \sin dx - \int \frac{\tan x}{\cos x} dx = -4\cos x + \int \frac{-\sin x}{(\cos x)^2} dx + C = -4\cos x + \int \frac{dy}{y^2} + C$

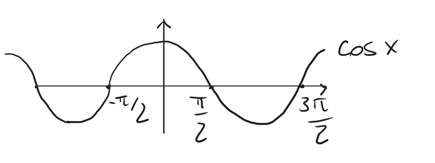
$y = \cos x$
$dy = -\sin x \, dx$

$= -4\cos x - \frac{1}{y} + C$

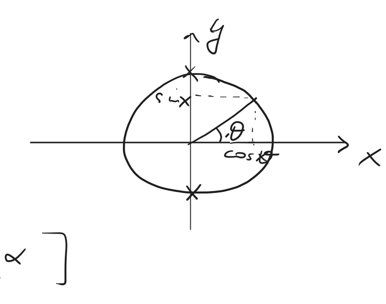
$= -4\cos x - \frac{1}{\cos x} + C$

$D(f) = \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$; $\int \alpha_1 f_1 + \alpha_2 f_2 = \alpha_1 \int f_1 + \alpha_2 \int f_2$

$f(x) = 4\sin x - \frac{\sin x}{(\cos x)^2}$



$\cos\theta = 0 \iff \theta = \frac{\pi}{2} + \pi k$, for some $k \in \mathbb{Z}$.



$\int x^\alpha \, dx = \begin{cases} \frac{x^{\alpha+1}}{\alpha+1} + C & \alpha \neq -1 \\ \ln|x| + C & \alpha = -1 \end{cases}$

$\left[\left(\frac{x^{\alpha+1}}{\alpha+1}\right)' = \frac{(\alpha+1)x^{\alpha+1-1}}{\alpha+1} = x^\alpha \right]$

$\left[(\ln|x|)' = \frac{1}{x} \right]$

1,b) $\int \frac{1-2x^2}{x^3} dx = \int \frac{1}{x^3} dx - \int \frac{2x^2}{x^3} dx = \int \frac{dx}{x^3} - \int \frac{2}{x} dx = \int x^{-3} dx - 2\int x^{-1} dx =$

$= \frac{x^{-2}}{-2} - 2 \log|x| + C = -\frac{1}{2x^2} - 2\log|x| + C$

2,a) $\lim_{M\to\infty} \int_0^M x e^{-x^2} dx = \lim_{M\to\infty} -\frac{1}{2}\int_0^M -2x \, e^{-x^2} dx = \lim_{M\to\infty} -\frac{1}{2}\int_0^{-M^2} e^y dy = \lim_{M\to\infty} \frac{1}{2}\int_{-M^2}^0 e^y dy$

$y = -x^2, \; dy = -2x\,dx$

$\int_0^M \underbrace{x}_{f'} \underbrace{e^{-x^2}}_{g} dx = \underbrace{\frac{x^2}{2}}_{f} \underbrace{e^{-x^2}}_{g} \Big]_0^M - \int_0^M \frac{x^2}{2}\left(-2xe^{-x^2}\right) dx$. complicated way.

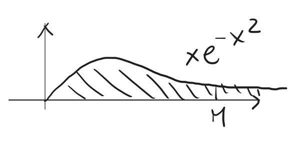
$= \lim_{M\to\infty} \frac{1}{2} e^y \Big]_{-M^2}^0$

$= \lim_{M\to\infty} \frac{1}{2}\left(1 - e^{-M^2}\right)$

$= \frac{1}{2}$

$\int e^x dx = e^x + C$

Geometric interpretation



2,b) $v_*(t) = \tanh(t/\sqrt{2})$; $v_*'(t) = \tanh'(t/\sqrt{2}) \cdot (t/\sqrt{2})' = \frac{1}{\sqrt{2}}\left(1 - \tanh^2(t/\sqrt{2})\right) = \frac{1}{\sqrt{2}} \frac{ch^2(t/\sqrt{2}) - sh^2(t/\sqrt{2})}{ch^2(t/\sqrt{2})} = \frac{1}{\sqrt{2}\,ch^2(\frac{t}{\sqrt{2}})}$

$\lim_{M\to\infty} \int_0^M (v_*'(t))^2 dt = \lim_{M\to\infty} \int_0^M \frac{dt}{2\,ch^4(t/\sqrt{2})} = \lim_{M\to\infty} \int_0^M \frac{dt}{\cancel{2}\left(\frac{e^{t/\sqrt{2}} + e^{-t/\sqrt{2}}}{2}\right)^4} = \lim_{M\to\infty} 8 \int_0^M \frac{dt}{\left(e^{t/\sqrt{2}} + e^{-t/\sqrt{2}}\right)^4}$

$t \xmapsto{f} \frac{t}{\sqrt{2}} = s \xmapsto{g} \tanh(s)$, $v_* = g \circ f$

$v_*'(t) = \frac{d}{dt} v_*(t) = g'(f(t)) \cdot f'(t) = \left(1 - th^2\left(\frac{t}{\sqrt{2}}\right)\right) \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}\,ch^2(t/\sqrt{2})}$

$g(s) = \tanh s \Rightarrow g'(s) = 1 - th^2(s)$; $s = g(t), \Rightarrow ds = g'(t)dt$

$s = t/\sqrt{2}$, $ds = \frac{1}{\sqrt{2}} dt$

$= \lim_{M\to\infty} 8 \int_0^{M/\sqrt{2}} \frac{\sqrt{2}\,ds}{(e^s + e^{-s})^4} = \lim_{M\to\infty} 8\sqrt{2} \int_0^M \frac{ds}{\left(e^{-s}(e^{2s}+1)\right)^4} = \lim_{M\to\infty} 8\sqrt{2} \int_0^M \frac{e^{4s} ds}{(e^{2s}+1)^4}$

$y = e^{2s}$

$dy = 2e^{2s} ds$

$= \lim_{M\to\infty} 8\sqrt{2} \int_0^M \frac{e^{2s} \cdot e^{2s} ds}{(e^{2s}+1)^4}$

$= \lim_{M\to\infty} \frac{8\sqrt{2}}{2} \int_1^{e^{2M}} \frac{y\,dy}{(y+1)^4} = \lim_{M\to\infty} 4\sqrt{2} \int_1^{e^{2M}} \overbrace{(y+1)^{-4}}^{f'} \underbrace{y}_{g} dy$

$= \lim_{M\to\infty} \frac{8\sqrt{2}}{2} \left(\frac{(y+1)^{-3}}{-3} y \Big]_1^{e^{2M}} - \int_1^{e^{2M}} \frac{(y+1)^{-3}}{-3} dy \right)$

$= \lim_{M\to\infty} 4\sqrt{2} \left(\frac{1}{(2)^3}\left(+\frac{1}{3}\right) + \frac{(e^{2M}+1)^{-3}}{-3} e^{2M} + \frac{1}{3} \int_1^{e^{2M}} (y+1)^{-3} dy \right)$

$= \lim_{M\to\infty} \left(\frac{\sqrt{2}}{2}\left(+\frac{1}{3}\right) - \frac{4\sqrt{2}}{3} \frac{e^{2M}}{(e^{2M}+1)^3} + \frac{4\sqrt{2}}{3}\left[\frac{(y+1)^{-2}}{-2} \right]_1^{e^{2M}} \right)$

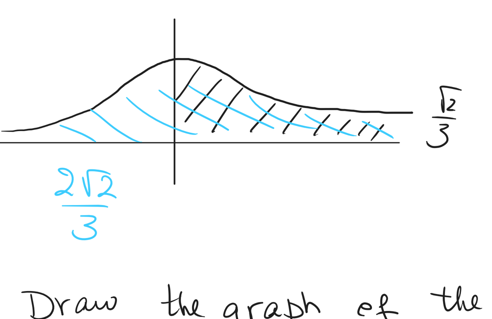
$= \lim_{M\to\infty} \frac{\sqrt{2}}{6} - \frac{4\sqrt{2}}{3} \frac{e^{2M}}{(e^{2M}+1)^3} + \frac{4\sqrt{2}}{3}\left(\frac{(e^{2M}+1)^{-2}}{-2} - \frac{(2)^{-2}}{-2} \right)$

$= \lim_{M\to\infty} \frac{\sqrt{2}}{6} - \frac{4\sqrt{2}}{3} \frac{e^{2M}}{(e^{2M}+1)^3} - \frac{4\sqrt{2}(e^{2M}+1)^{-1}}{6} + \frac{4\sqrt{2}}{6} \cdot \frac{1}{4}$

$= \lim_{M\to\infty} \left(\frac{\sqrt{2}}{3} - \frac{4\sqrt{2}}{3} \frac{e^{2M}}{(e^{2M}+1)^3} - \frac{2\sqrt{2}}{3} \frac{1}{e^{2M}+1} \right)$

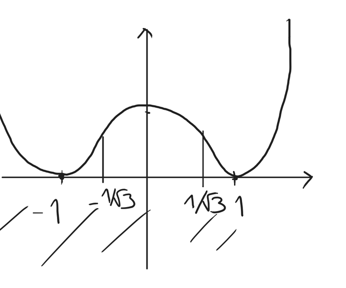
$= \lim_{M\to\infty} \frac{\sqrt{2}}{3}\left(1 - 2\frac{e^{2M}}{(e^{2M}+1)^3} - \underbrace{\frac{2}{e^{2M}+1}}_{\to 0 \; M\to\infty}\right) = \frac{\sqrt{2}}{3}$

$\frac{e^{2M}}{(e^{2M}+1)^3} = \frac{e^{2M}}{e^{6M}(1+e^{-2M})^3} = \frac{1}{e^{4M}(1+e^{-2M})^3} \to 0 \quad M\to\infty$



3,a) Draw the graph of the function $f(x) = \frac{(1-x^2)^2}{4}$, specifying the limits, the monotonicity and the convexity (or concavity) of f.

- $D(f) = \mathbb{R}$.
- Zeros of f. $f(x) = 0 \iff 1 - x^2 = 0 \iff x = 1$ or $x = -1$.
- Sign of f. $f(x) > 0 \iff x \neq \pm 1$



- $\lim_{x\to+\infty} f(x) = +\infty$
- $\lim_{x\to-\infty} f(x) = +\infty$
- f is even

- critical points: $f'(x) = 0 \iff x = -1 \lor x = 0 \lor x = 1$

$f'(x) = \left(\frac{(1-x^2)^2}{4}\right)' = \frac{1}{4} 2(1-x^2)(-2x) = -x(1-x^2) = x^3 - x$

$f'(x) = 0 \iff x(x^2-1) = 0 \iff x = 0 \lor x = 1 \lor x = -1.$

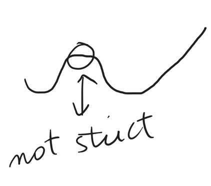
- Sign of the derivative. $f'(x) > 0 \iff \underbrace{x(x^2-1)}_{= x(x-1)(x+1)} > 0$

		-1		0		1	
$x-1$	−		−		−		+
x	−		−		+		+
$x+1$	−		+		+		+
	−		+		−		+

$f'(x) < 0 \iff x \in (-\infty, -1) \cup (0, 1)$

$f'(x) > 0 \iff x \in (-1, 0) \cup (1, +\infty)$

± 1 is a strict global minimizer; 0 is a strict local maximizer.



loc. max, not strict

- Concavity - convexity:

$f'' = 3x^2 - 1$

- f is convex $\iff 3x^2 - 1 > 0 \iff x > \frac{1}{\sqrt{3}} \lor x < -\frac{1}{\sqrt{3}}$
- f is concave $\iff 3x^2 - 1 < 0 \iff -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

3,c) Draw the graph of
$f(x) = e^{\frac{x-1}{x-3}}$

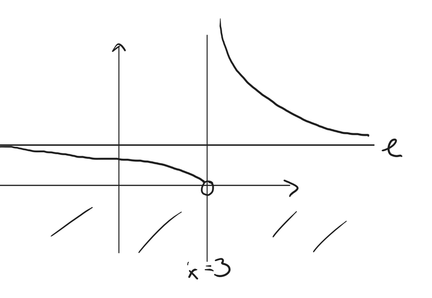
- $D(f) = \mathbb{R} \setminus \{3\}$
- Zeros of f. f never vanishes.
- Sign of f. $f(x) > 0 \quad \forall x \in D(f)$

$\lim_{x\to\infty} e^{\frac{x-1}{x-3}} = e^1 = e$

$\lim_{x\to-\infty} e^{\frac{x-1}{x-3}} = e$

$\lim_{x\to3^+} e^{\frac{x-1}{x-3}} = e^{\frac{2}{0^+}} = +\infty$

$\lim_{x\to3^-} e^{\frac{x-1}{x-3}} = e^{\frac{2}{0^-}} = e^{-\infty} = 0^+$



- Monotonicity:

$f'(x) = \left(e^{\frac{x-1}{x-3}}\right)' = e^{\frac{x-1}{x-3}} \left(\frac{x-1}{x-3}\right)' = e^{\frac{x-1}{x-3}} \left(\frac{1}{x-3} - \frac{x-1}{(x-3)^2} \right)$

$= f(x) \frac{x-3-(x-1)}{(x-3)^2} = \frac{-2}{(x-3)^2} f(x) < 0$

Merry Christmas