

Functions: let X, Y be sets. Then a function $f: X \rightarrow Y$ is a law which, to any element $x \in X$, associates a unique element $y \in Y$.

$X = \{\text{students in this class}\}$

$Y = \{\text{letters of the alphabet}\}$

$f: X \rightarrow Y$

$x \in X \mapsto \text{first letter of its name}$

$f(\text{Giorgio}) = G$. $f(\text{Anastasia}) = A$.

Definition: we say that the set X is the domain of f .

$X = \mathbb{R}$, $f: \mathbb{R} \rightarrow \mathbb{R}$ ($Y = \mathbb{R}$)

$x \in \mathbb{R} \mapsto 2x \in \mathbb{R}$

the domain of f , also denoted by $D(f)$, is $\mathbb{R}$

Definition: the image of a function is the set

$$f(X) = \{f(x) : x \in X\}$$

the image of a point $x \in X$ is $f(x)$.

Remark: $f(x) \in Y$, but it is not always true that $f(x) = Y$.

$X = \mathbb{R}$, $Y = \mathbb{R}$, $f: \mathbb{R} \rightarrow \mathbb{R}$

$f(x) = x^2$

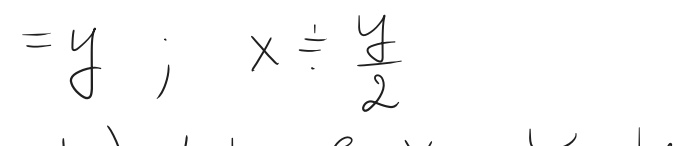
$f(X) = [0, +\infty)$, because $x^2 \geq 0, \forall x \in \mathbb{R}$.

Definition: the graph of a function is the set

$$\Gamma(f) = \{(x, f(x)) : x \in D(f)\} = \{(x, f(x)) : x \in X\}$$

$D(f) = X$

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$



Remark: let $f: X \rightarrow Y$, then $\Gamma(f) \subseteq X \times Y$, where $X \times Y = \{(x, y) : x \in X, y \in Y\}$ is the cartesian product of X and Y .

$X = Y = \mathbb{R} \Rightarrow X \times Y = \mathbb{R}^2$ is the cartesian plane.

example: $X = Y = \mathbb{R}$, $f(x) = 2x$, what is $f(\mathbb{R}) = ?$

$f(\mathbb{R}) = (-\infty, +\infty) = \mathbb{R}$

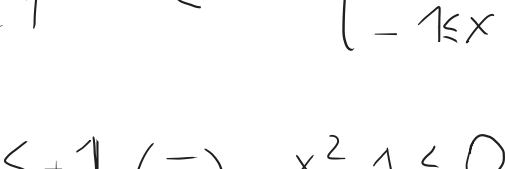
$\forall y \in Y, \exists x \in X : f(x) = y$

$f(\mathbb{R}) = \{2x : x \in \mathbb{R}\} = \mathbb{R}$

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : 2x = y, x = \frac{y}{2}$

Def: (Inverse image of a set) let $f: X \rightarrow Y$ be a function, let $Z \subseteq Y$.

Then the set $f^{-1}(Z) = \{x \in X : f(x) \in Z\}$



$$f^{-1}(Z) = [a, x_1] \cup \{x_2\} \cup [x_3, b]$$

$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$



$$f^{-1}([-1, 1]) = \{x \in \mathbb{R} : -1 \leq x^2 \leq 1\} = \{x \in \mathbb{R} : x^2 \geq -1 \wedge x^2 \leq 1\}$$

$$= [-1, 1]$$

$$\begin{cases} x^2 \geq -1 \\ x^2 \leq 1 \end{cases} \Leftrightarrow \begin{cases} \forall x \in \mathbb{R} \\ -1 \leq x \leq 1 \end{cases} \Leftrightarrow -1 \leq x \leq 1$$

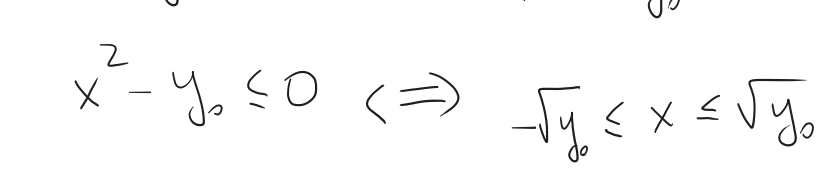


$$x^2 \leq 1 \Leftrightarrow x^2 - 1 \leq 0$$

$$x^2 - 1 = 0 \Leftrightarrow x = \pm 1$$

$$x^2 - 1 \geq 0 \Leftrightarrow x \geq 1 \vee x \leq -1$$

• $y_0 \in \mathbb{R}, y_0 > 0$: we want to solve $x^2 - y_0 \geq 0 \Leftrightarrow x^2 \geq y_0$



$$x^2 = y_0 \Leftrightarrow x = \pm \sqrt{y_0}$$

when is the parabola above the straight line $y = y_0$?

$$x \geq \sqrt{y_0} \vee x \leq -\sqrt{y_0}$$

$$x^2 - y_0 \geq 0 \Leftrightarrow x \geq \sqrt{y_0} \vee x \leq -\sqrt{y_0}$$

$$x^2 - y_0 \leq 0 \Leftrightarrow -\sqrt{y_0} \leq x \leq \sqrt{y_0}$$

Equivalent way: $y = x^2 - y_0$

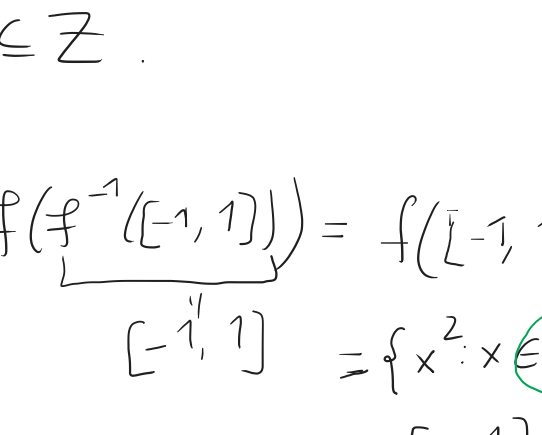


if $y_0 < 0 \Rightarrow x^2 - y_0 > 0 \quad \forall x \in \mathbb{R}$

$x^2 = y_0$: when is this eq solved? Never

$\Rightarrow$ the parabola $y = x^2 - y_0$ does not intersect the x axis

$$x^2 = 0 \Rightarrow y = y_0 > 0$$



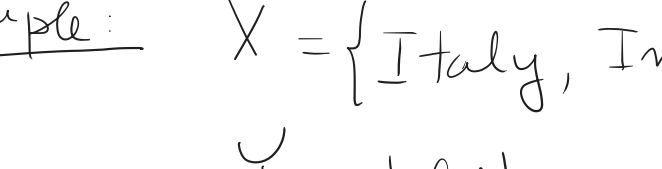
Remark: let $f: X \rightarrow Y$ be a function, let $Z \subseteq Y$. Then $f(f^{-1}(Z)) \subseteq Z$.

if $x \in f^{-1}(Z) \Rightarrow f(x) \in Z \Rightarrow f(f^{-1}(Z)) \subseteq Z$.

$X = Y = \mathbb{R}$, $f(x) = x^2$, $f^{-1}([-1, 1]) = [-1, 1] \Rightarrow f(f^{-1}([-1, 1])) = f([-1, 1])$

$$[-1, 1] = \{x^2 : x \in [-1, 1]\}$$

$$= [0, 1] \subseteq [-1, 1]$$



It is not always true that $f(f^{-1}(Z)) = Z$.

Remark: let $W \subseteq X$, let $f: X \rightarrow Y$. Then $W \subseteq f^{-1}(f(W))$.

$$\forall x \in W, f(x) \in f(W) \Leftrightarrow \forall x \in W, x \in f^{-1}(f(W)) \Leftrightarrow W \subseteq f^{-1}(f(W))$$

Example: $X = \{\text{Italy, Indonesia, France, Portugal}\}$

$Y = \{\text{letters of the alphabet}\}$

$f: X \rightarrow Y$

$x \in X \mapsto f(x) = \text{initial letter of } x$

$f(\text{Italy}) = I, f(\text{Indonesia}) = I, f(\text{France}) = F, f(\text{Portugal}) = P$

$f^{-1}(I) = \{\text{Italy, Indonesia}\}, f^{-1}(F) = \{\text{France}\}, f^{-1}(P) = \{\text{Portugal}\}$

$Z = \{A, I\} \subseteq Y \Rightarrow f(f^{-1}(Z)) = f(\{I\}) = \{I\}$

$$\Rightarrow f(f^{-1}(Z)) \subsetneq Z$$

$Z' = \{I\} \Rightarrow f(f^{-1}(Z')) = Z'$

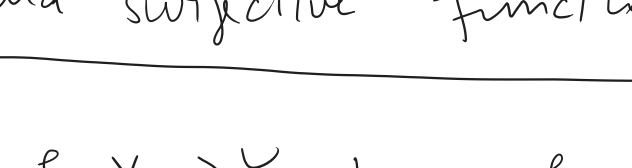
• $W = \{\text{Italy}\} \subseteq X \Rightarrow f(W) = \{I\} \Rightarrow f^{-1}(f(W)) = \{\text{Italy, Indonesia}\}$

$W \subsetneq f^{-1}(f(W))$.

• $W' = \{\text{Italy, France}\} \Rightarrow f(W') = \{I, F\} \Rightarrow f^{-1}(f(W')) = \{\text{Italy, Indonesia, France}\}$

$W' \subsetneq f^{-1}(f(W'))$.

• $f: \mathbb{R} \rightarrow \mathbb{R}^2, f(t) = x_0 + t v, v \in \mathbb{R}^2 \setminus \{0\}$



$$f(\mathbb{R}) = L = \{x_0 + t v : t \in \mathbb{R}\} \subsetneq \mathbb{R}^2$$

Injective and surjective functions

Def: let $f: X \rightarrow Y$ be a function. We say that

• f is injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \forall x_1, x_2 \in X$

• f is surjective if $f(X) = Y$.

Remark: • f is injective $\Leftrightarrow \forall x_1, x_2 \in X, x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$ (different inputs $\Rightarrow$ different outputs)

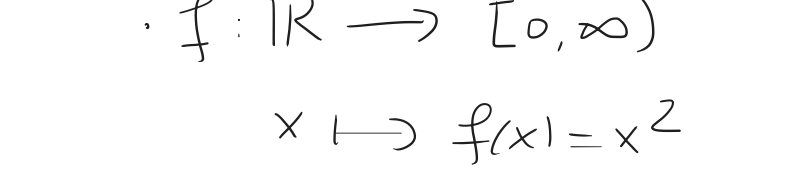
• f is surjective $\Leftrightarrow \forall y \in Y, \exists x \in X : f(x) = y$.

Examples: $f(x) = x^2, X = Y = \mathbb{R}$.

Is f injective? $f(1) = 1, f(-1) = 1 \quad \underline{\text{No}}$

Is f surjective? $\forall y \in Y = \mathbb{R}, \exists x \in X = \mathbb{R} : f(x) = x^2 = y$

is it true? $f(\mathbb{R}) = [0, \infty) \subsetneq \mathbb{R} = Y$.



$y_0 < 0 \Rightarrow y_0 \notin f(\mathbb{R}) \Rightarrow f$ is not surjective.

$y_0 > 0 \Rightarrow$ the straight line $y = y_0$ intersects the graph more than once $\Rightarrow f$ is not injective.

• $f: \mathbb{R} \rightarrow [0, \infty)$

$x \mapsto f(x) = x^2$

Is f surjective? yes.

let $f: X \rightarrow Y$, then $f: X \rightarrow f(X)$ is surjective.

Definition: (Restriction of a function). let $f: X \rightarrow Y$ be a function, let $A \subseteq X$ be a subset. Then we set $f|_A: A \rightarrow Y$

$$x \in A \mapsto f(x) \in Y$$

if $f: \mathbb{R} \rightarrow [0, \infty)$, then $f|_{[0, \infty)}: [0, \infty) \rightarrow [0, \infty)$ is injective.

$$x \mapsto x^2$$

$$x_1^2 = x_2^2 = y \geq 0 \text{ and } x_1 \geq 0, x_2 \geq 0 \Rightarrow x_1 = x_2$$

$x^2 = y \Rightarrow x = \pm \sqrt{y}, y \in [0, \infty)$ comes from 2 possible inputs, that is $\sqrt{y}$ or $-\sqrt{y}$.



the only input in $[0, \infty)$ is $\sqrt{y}$.

• $X = \{\text{Indonesia, Italy, France, Portugal}\}, Y = \{\text{letters of the alphabet}\}$

$f: X \rightarrow Y$

$x \in X \mapsto \text{first letter of } x$

Is f injective? No. $f(\text{Italy}) = f(\text{Indonesia}) = I$

is f surjective? No: $a \in Y$, but $\nexists x \in X : f(x) = a$.

• $f: \mathbb{R} \rightarrow \mathbb{R}$

$x \mapsto 2x + 1 = y$

Is f surjective? Is it true that $\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : 2x + 1 = y$?

$$2x + 1 = y \Leftrightarrow 2x = y - 1 \Leftrightarrow x = \frac{y-1}{2} \in \mathbb{R}$$



Take $y \in Y$, then f is surjective if the equation $f(x) = y$ admits at least a solution $x \in X$.

• f is injective if such a solution is unique (if it exists).

$f: \mathbb{R} \rightarrow \mathbb{R}$

$f(x) = 2x + 1$ is also injective.

Definition: A function $f: X \rightarrow Y$ is bijective if it is injective and surjective.

For example the function $f: \mathbb{R} \rightarrow \mathbb{R}$ is bijective.

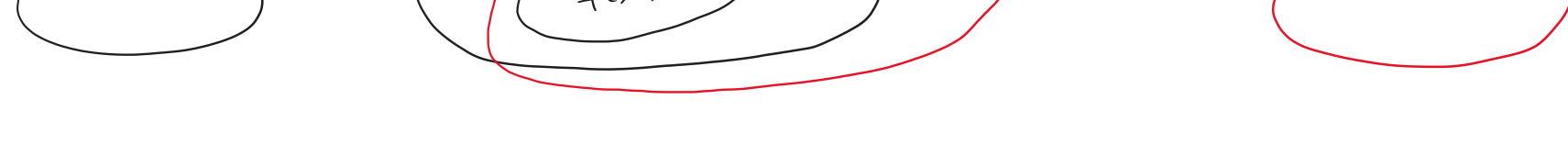
the function $f: \mathbb{R} \rightarrow \mathbb{R}$ is bijective.

$$x \mapsto 3x$$

Definition (Composition of functions). let $f: X \rightarrow Y$ and $g: Z \rightarrow W$ be functions. Assume that $f(X) \subseteq Z$. Then the function

$$g \circ f: X \rightarrow W$$

defined by $g \circ f(x) = g(f(x))$ is well defined, and it is known as "g composed with f".



$f: \mathbb{R} \rightarrow \mathbb{R}, g: [0, \infty) \rightarrow \mathbb{R}$

$f(x) = x + 1, g(y) = \sqrt{y}$

$g \circ f: \mathbb{R} \rightarrow \mathbb{R}$

$x \mapsto \sqrt{x+1}$

$x \in [-1, \infty) \Rightarrow f(x) = x + 1 \in [0, \infty) = D(g) \Rightarrow$ we can define $g(f(x))$

and $g(f(x)) = \sqrt{x+1}$.

Definition: let X be a set, $\text{Id}_X: X \rightarrow X$

$$x \mapsto x$$

Definition (Inverse function) Let X, Y be sets, let $f: X \rightarrow Y$ be a function. We say that f is invertible if

$\exists$ a function $g: Y \rightarrow X$ s.t. $f \circ g = \text{Id}_Y$ and $g \circ f = \text{Id}_X$.

This function g is known as the inverse of f .