Powers and roots

$x \in \mathbb{R}, n \in \mathbb{Z}$

$$x^n := \begin{cases} \underbrace{x \cdots x}_{n \text{ times}} & n > 0 \\ 0 & n = 0,\ x \neq 0 \\ \frac{1}{x^{-n}} & n < 0,\ x \neq 0 \end{cases}$$

$x = \frac{2}{3}, n = -2, \quad x^n = \frac{9}{4} \ldots$

Roots: let $y \geq 0$, let $n \in \mathbb{N} \setminus \{0\}$, $\sqrt[2n]{y} := \sup\{x \in \mathbb{R} : 0 \leq x^{2n} \leq y\}$

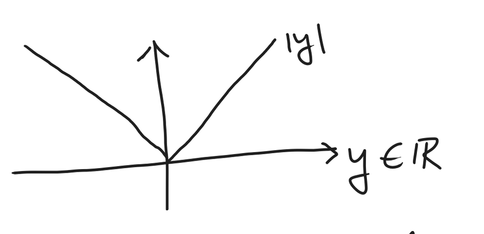
$n = 1$: $\sqrt[2]{y} =: \sqrt{y}$.

properties of even roots: $\cdot\ \forall y \geq 0, \forall n \in \mathbb{N} \setminus \{0\}, \left(\sqrt[2n]{y}\right)^{2n} = y$

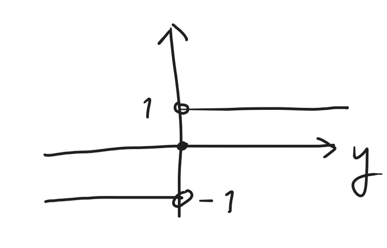
$\cdot\ \forall x \geq 0, \forall n \in \mathbb{N} \setminus \{0\}, \sqrt[2n]{x^{2n}} = x$

$x = -2, n = 2, \quad x^2 = 4 \iff (-2)^2 = 4$

- It follows from the definition that $\sqrt[2n]{y} \geq 0, \forall y \geq 0$ and $\sqrt[2n]{y} = 0 \iff y = 0$
- $\sqrt{4} = 2 \neq -2$.
- $y \in \mathbb{R}$. Then we set $|y| := \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases}$



- $y \in \mathbb{R}$, $\operatorname{sgn}(y) := \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -1 & \text{if } y < 0 \end{cases}$



As a consequence, $\forall y \in \mathbb{R}$, $y = \operatorname{sgn}(y)|y|$

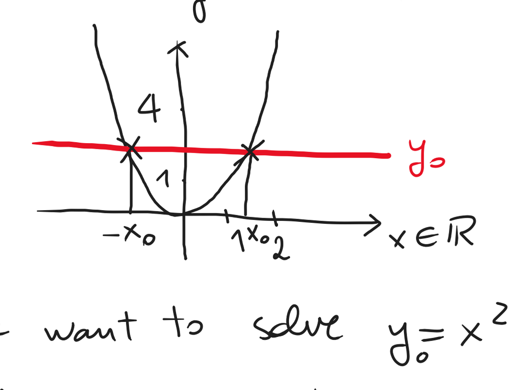
Claim: $\forall x \in \mathbb{R}, \forall n \in \mathbb{N} \setminus \{0\}, \sqrt[2n]{x^{2n}} = |x|$.

if $x < 0 \Rightarrow x^{2n} > 0 \Rightarrow$ it makes sense to compute $\sqrt[2n]{x^{2n}}$

example: $x = -2 \Rightarrow x^2 = 4 \Rightarrow \sqrt{x^2} = \sqrt{4} = 2 = |-2|$.

Remark: Let $y > 0$. Then equation $x^{2n} = y$ has exactly 2 solutions, given by $\pm\sqrt[2n]{y}$.

We want to solve equation $x^2 = 4$. This equation has exactly 2 solutions given by ± 2.



$y = x^2$

$0 \to 0$
$1 \to 1 \qquad -1 \to 1$
$2 \to 4 \qquad -2 \to 4$
$3 \to 9$

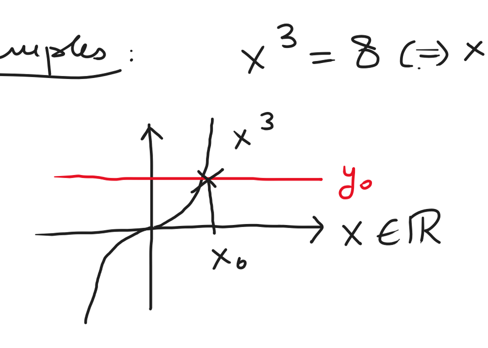
$(-x)^2 = x^2$

we want to solve $y_0 = x^2$

Odd roots: $n \in \mathbb{N}$, $y \in \mathbb{R}$. $\sqrt[2n+1]{y} := \operatorname{sgn}(y) \sup\{x \in \mathbb{R} : 0 \leq x^{2n+1} \leq |y|\}$

Remark: $\forall y \in \mathbb{R}$, equation $x^{2n+1} = y$ has a unique solution, given by $\sqrt[2n+1]{y}$.

examples: $x^3 = 8 \iff x = 2$; $x^3 = -8 \iff x = -2$



Rational powers: Let $y \geq 0$, $m \in \mathbb{Z}$, $n \in \mathbb{N} \setminus \{0\}$.

$$y^{\frac{m}{n}} := \begin{cases} \sqrt[n]{y^m} & y > 0 \\ 0 & y = 0 \end{cases}$$

Let $y < 0$, let $n \in \mathbb{N}$. $y^{\frac{m}{2n+1}} = \sqrt[2n+1]{y^m}$.

When defining $y^{\frac{m}{n}}$, we can always assume that m and n have no common factors. In particular, we can assume that one of the 2 is odd.
If m is odd and n is even, we are not allowed to define $y^{\frac{m}{n}}$ for $y < 0$. (because $y^m < 0$)

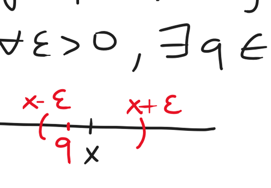
$4^{3/2} = \sqrt[2]{4^3} = \sqrt[2]{64} = 8$

$4 = 2^2 \Rightarrow 4^3 = (2^2)^3 = 2^{2 \cdot 3} = 2^6 \Rightarrow \sqrt[2]{4^3} = \sqrt[2]{2^6} = 2^{6/2} = 2^3 = 8$

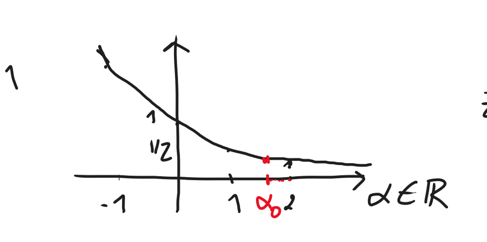
Real powers: Let $y \geq 0$, let $\alpha \in \mathbb{R}$.

$$y^\alpha := \begin{cases} 0 & y = 0 \\ \sup\{y^q : q \in \mathbb{Q}, q > \alpha\} & 0 < y < 1 \\ 1 & y = 1 \\ \sup\{y^q : q \in \mathbb{Q}, q < \alpha\} & y > 1 \end{cases}$$

Density of $\mathbb{Q}$ in $\mathbb{R}$: $\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists q \in \mathbb{Q} : x - \varepsilon < q < x + \varepsilon$

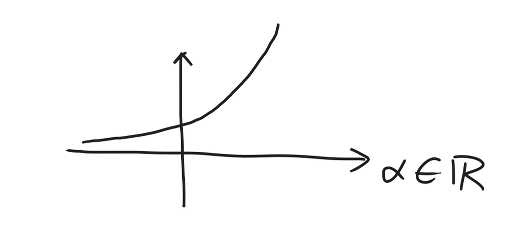


$0 < y < 1$



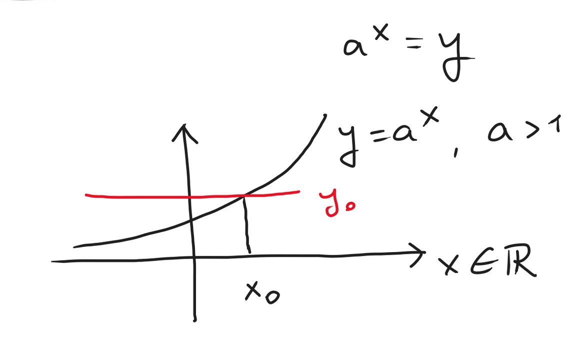
$z = y^\alpha \qquad [y = \frac{1}{2}] \quad (\frac{1}{2})^{-1} = \frac{1}{1/2} = 2$

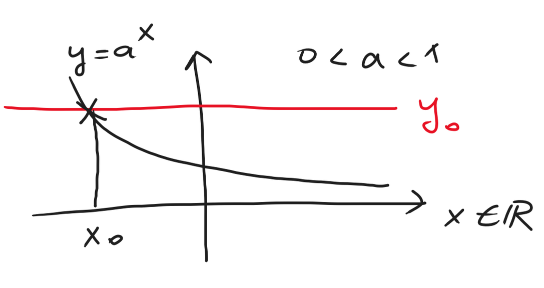
$y > 1$



$y = 2 \qquad \alpha = 0 \to 2^0 = 1$
$\alpha = 1 \to 2^1 = 2$
$\alpha = 2 \to 4$

Logarithms: $a > 0, a \neq 1$. $\forall y > 0, \exists!$ solution x to equation $a^x = y$





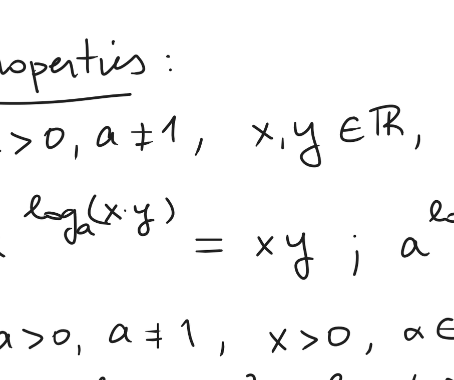
Definition: $\forall a > 0, a \neq 1, \forall y > 0$, we call the unique solution x to equation $a^x = y$ $\quad x = \log_a y$

$\boxed{x = \log_a y \iff a^x = y}$

- $a^{\log_a y} = y \quad \forall y > 0$
- $\log_a(a^x) = x \quad \forall x \in \mathbb{R}$.

$2^3 = 8 \iff 3 = \log_2 8$

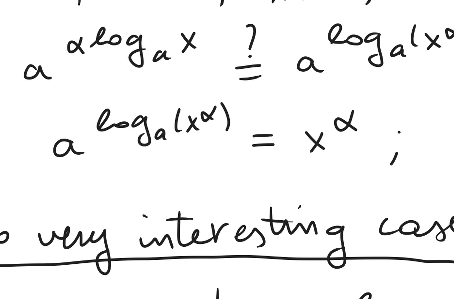
- $\forall a > 0, a \neq 1, \log_a 1 = 0 \iff a^0 = 1$



$a > 1$

$\log_a a^2 = 2 \qquad \log_a a^{-3} = -3$

$\log_a a^{-2} = -2$



$0 < a < 1$

properties:

- $a > 0, a \neq 1$, $x, y \in \mathbb{R}$, $x > 0$, $y > 0$, $\log_a(x \cdot y) = \log_a x + \log_a y$

$a^{\log_a(x \cdot y)} = xy$; $a^{\log_a x + \log_a y} = a^{\log_a x} \cdot a^{\log_a y} = xy$

- $a > 0, a \neq 1, x > 0, \alpha \in \mathbb{R} \Rightarrow \alpha \log_a x = \log_a(x^\alpha)$

$a^{\alpha \log_a x} \stackrel{?}{=} a^{\log_a(x^\alpha)}$

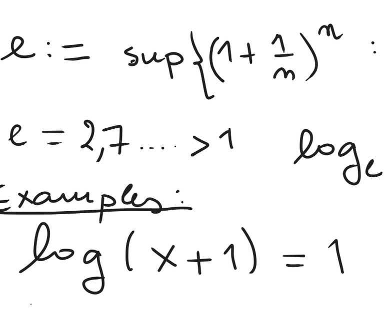
$a^{\log_a(x^\alpha)} = x^\alpha$; $a^{\alpha \log_a x} = (a^{\log_a x})^\alpha = x^\alpha$

two very interesting cases:

- $a = 10$, $\text{Log} = \log_{10}$.

Earthquakes: A = amplitude of the oscillation.

$M = \text{Log}\, A - \text{Log}\, A_0 \qquad [A_0 = 1]$



$A_2 = 10^2 \qquad A_4 = 10^4$

$\frac{A_4}{A_2} = \frac{10^4}{10^2} = 10^{4-2} = 100$

PH: We have a solution (water and salt, water and ...)
We need to measure the concentration of hidrogene ions. $\rightsquigarrow$ pH $= -\text{Log}$(concentration)
x = concentration, $\quad$ pH $= -\text{Log}\, x$

- $e := \sup\{(1 + \frac{1}{n})^n : n \in \mathbb{N} \setminus \{0\}\}$.

$e = 2{,}7 \ldots > 1 \qquad \log_e =: \log = \ln$

Examples:

- $\log(x + 1) = 1 \qquad x + 1 = e \qquad [a^{\log_a x} = x]$

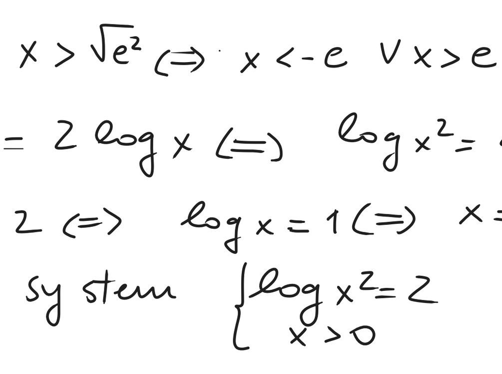
e.c. $x + 1 > 0 \iff x > -1 \qquad x = e - 1 > -1$

$\cancel{e}^{\cancel{\log}(x+1)} = e^1$

- $\underbrace{\log(x^2)}_{a} > \underbrace{2}_{b}$ $\quad \cdot\ a < b \iff e^a < e^b$

$\cancel{e}^{\cancel{\log}(x^2)} > e^2$

$\boxed{x^2 > e^2}$



$x^2 = y_0$

$x_\pm = \pm\sqrt{y_0}$

- $x^2 > y_0 \iff x < -\sqrt{y_0} \vee x > \sqrt{y_0}$
- $x^2 < y_0 \iff -\sqrt{y_0} < x < \sqrt{y_0}$

$x^2 > e^2 \iff x < -\sqrt{e^2} \vee x > \sqrt{e^2} \iff x < -e \vee x > e$

example $\quad \begin{cases} \log x^2 = 2 \log x \iff \log x^2 = 2 \\ x > 0 \end{cases}$

$2 \log x = 2 \iff \log x = 1 \iff x = e$

here we solved the system $\begin{cases} \log x^2 = 2 \\ x > 0 \end{cases}$