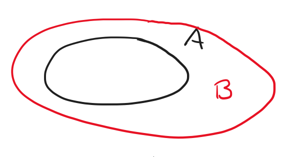
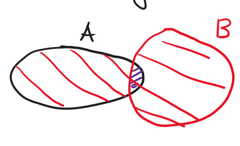
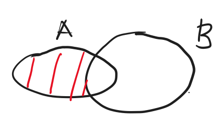
Let A, B be set. We say that $A \subseteq B$ if $\forall x \in A \Rightarrow x \in B$.



· Let A, B be two set.
- We say that $x \in A \cap B$ if $x \in A$ and $x \in B$
- We say that $x \in A \cup B$ if $x \in A$ or $x \in B$.
- We say that $x \in A \setminus B$ if $x \in A$ and $x \notin B$

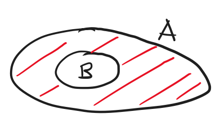


$A \cap B$, $A \cup B$



$A \setminus B$

· If $B \subseteq A$, then $A \setminus B$ is known as the complement of B in A, and it is denoted by $\complement_A(B)$ (or B^c)



$A \setminus B = B^c = \complement_A(B) = \{x \in A : x \notin B\}$

· $A = \mathbb{Z}$, $B = \mathbb{N}$, What is $A \setminus B$? (or $\mathbb{N}^c$)

$A \setminus B = \mathbb{Z} \setminus \mathbb{N} = \mathbb{N}^c = \{0, \pm 1, \pm 2 \dots\} \setminus \{0, 1, 2 \dots\}$
$= \{-1, -2, -3 \dots\}$

$A = \{a, b, c, d\}$, $B = \{a, b\}$, $A \setminus B = \{c, d\}$

· $A = \mathbb{R}$, $B = (0, \infty) = \{x \in \mathbb{R} : x > 0\}$

$A \setminus B = \mathbb{R} \setminus (0, \infty) = (-\infty, 0] = \{x \in \mathbb{R} : x \leq 0\}$

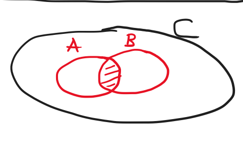
$A = \mathbb{R}$, $B = [a, b)$, $a < b$

$A \setminus B = ?\ (-\infty, a) \cup [b, \infty)$

$\mathbb{R} \setminus [a,b) = \{x \in \mathbb{R} : x \notin [a,b)\} = \{x \in \mathbb{R} : \neg(a \leq x < b)\}$
$= \{x \in \mathbb{R} : \neg(x \geq a \underset{\text{and}}{\wedge} x < b)\} = \{x \in \mathbb{R} : x < a \vee x \geq b\}$
$= \{x \in \mathbb{R} : x \in (-\infty, a) \vee x \in [b, \infty)\} = (-\infty, a) \cup [b, \infty)$

<u>Remember</u>: Let X and Y be proposition: Then
$\neg(X \wedge Y) = (\neg X) \vee (\neg Y)$
$\neg(X \vee Y) = (\neg X) \wedge (\neg Y)$

<u>As a consequence</u>: Let A, B, C be sets, $A \subseteq C$, $B \subseteq C$



· $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$
· $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$

$C \setminus (A \cap B) = \{x \in C : x \notin A \cap B\} = \{x \in C : \neg(x \in A \wedge x \in B)\}$
$= \{x \in C : x \notin A \vee x \notin B\} = \{x \in C : (x \in C \setminus A) \vee x \in (C \setminus B)\}$
$= (C \setminus A) \cup (C \setminus B)$.

<u>Examples</u>: $A = (-\infty, 1)$, $B = (0, \infty)$

$A \cap B = (0, 1)$

$\mathbb{R} \setminus (A \cap B) = (-\infty, 0] \cup [1, \infty)$
$\mathbb{R} \setminus A = [1, \infty)$; $\mathbb{R} \setminus B = (-\infty, 0]$
$\Rightarrow \mathbb{R} \setminus (A \cap B) = (\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B)$

$C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$

$C \setminus (A \cup B) = \{x \in C : x \notin A \cup B\} = \{x \in C : \neg(x \in A \cup B)\} =$
$= \{x \in C : \neg(x \in A \vee x \in B)\} = \{x \in C : (x \notin A) \wedge (x \notin B)\} =$
$= \{x \in C : x \in C \setminus A \wedge x \in C \setminus B\} = (C \setminus A) \cap (C \setminus B)$

<u>Example</u>: $A = \{0\}$, $B = [1, 2)$

$C \setminus (A \cup B) = (-\infty, 0) \cup (0, 1) \cup [2, \infty) = \{x \in \mathbb{R} : x \notin A \wedge x \notin B\} =$
$= \{x \in \mathbb{R} : x \neq 0 \wedge \neg(1 \leq x < 2)\} = \{x \in \mathbb{R} : x \neq 0 \wedge (x < 1 \vee x \geq 2)\}$
$= ((-\infty, 1) \cup [2, \infty)) \setminus \{0\} = (-\infty, 0) \cup (0, 1) \cup [2, \infty)$

$C \setminus A = (-\infty, 0) \cup (0, \infty)$; $C \setminus B = (-\infty, 1) \cup [2, \infty)$

$\Rightarrow (C \setminus A) \cap (C \setminus B) = (-\infty, 0) \cup (0, 1) \cup [2, \infty)$

<u>The extrema of a set</u>:

Let $A \subseteq \mathbb{R}$ be a set.

<u>Def</u>:
- we say that $x \in A$ is the maximum of A if $x \geq a$, $\forall a \in A$.
- we say that $x \in A$ is the minimum of A if $x \leq a$, $\forall a \in A$.

<u>Examples</u>:
- $I = [0, 1] \Rightarrow \min I = 0$, $\max I = 1$
- $A = \{0, 1, \frac{3}{2}, 4\} \Rightarrow \min A = 0$, $\max A = 4$

<u>Remark</u>: The maximum and the minimum of a set do not always exist.

<u>Example</u>: · $I = (0, 1) \Rightarrow \nexists \max I, \min I$.

$x \in I \Rightarrow \exists y \in \mathbb{R} : x < y < 1$, $\exists z \in \mathbb{R} : 0 < z < x$.

$y = \frac{x+1}{2}$ $\quad z = \frac{x}{2}$

- $I = [a, b)$. $\nexists \max I$. $\min I = a$.
- $I = (a, b]$, $\nexists \min I$, $\max I = b$
- $I = [a, \infty)$, $\min I = a$, $\nexists \max I$.

· Let $A \subseteq \mathbb{R}$ be a set. Then we define the upper set of A as $M(A) := \{x \in \mathbb{R} : x \geq a, \forall a \in A\}$
and the lower set of A as
$m(A) := \{x \in \mathbb{R} : x \leq a, \forall a \in A\}$

A

<u>Theorem</u>:
- if $M(A) \neq \emptyset \Rightarrow M(A)$ has a minimum
- if $m(A) \neq \emptyset \Rightarrow m(A)$ has a maximum.

(this result can be shown by using the continuity property of $\mathbb{R}$)

<u>Definition</u>: Let $A \subseteq \mathbb{R}$ be a set. Then we set

$$\sup(A) = \begin{cases} +\infty & \text{if } M(A) = \emptyset \\ \min M(A) & \text{if } M(A) \neq \emptyset \end{cases}$$

$$\inf(A) = \begin{cases} -\infty & \text{if } m(A) = \emptyset \\ \max m(A) & \text{if } m(A) \neq \emptyset. \end{cases}$$

The inf and the sup are always well defined, $\forall A \subseteq \mathbb{R}$.

· $I = (0, 1) \Rightarrow \inf I = 0$, $\sup I = 1$

$M(I) = [1, \infty) \Rightarrow \min M(I) = 1 \Rightarrow \sup I = 1$

$m(I) = (-\infty, 0] \Rightarrow \max m(I) = 0 \Rightarrow \inf I = 0$.

· $I = [0, \infty) \Rightarrow \sup I = +\infty$
$\inf I = 0$
$\min I = 0$

· <u>Remark</u>: Let $A \subseteq \mathbb{R}$. If A has a max $\Rightarrow \max A = \sup A$
If A has a min $\Rightarrow \min A = \inf A$
And in these cases, we have $\sup A \in \mathbb{R}$ (or $\inf A \in \mathbb{R}$).

· <u>Definition</u>: Let $A \subseteq \mathbb{R}$ be a set. Then we say that
- A is bounded above if $\sup A \in \mathbb{R}$
- A is bounded below if $\inf A \in \mathbb{R}$

<u>Example</u>:
- $I = (0, \infty)$ is not bounded, but I is bounded below, because $\inf I = 0 \in \mathbb{R}$.
- $I = (-\infty, 0]$ is not bounded, but I is bounded above, because $\sup I = 0 \in \mathbb{R}$.

<u>Remark</u>: $A \subseteq \mathbb{R}$ is bounded $\Leftrightarrow A$ is both bounded below and bounded above

<u>proof</u>:
· A bounded $\Rightarrow \exists a, b \in \mathbb{R} : A \subset (a, b) \Rightarrow a \leq \inf A \leq \sup A \leq b$
$\Rightarrow \inf A, \sup A \in \mathbb{R}$
· A is bounded above and below $\Rightarrow \inf A, \sup A \in \mathbb{R}$
and $A \subset (\inf A - 1, \sup A + 1) \Rightarrow A$ is bounded.

Note that $A \not\subseteq (\inf A, \sup A)$. (in general). $A = [0, 1] \not\subseteq (0, 1)$.

· If $A \subseteq (a, b) \underset{\not\Leftarrow}{\Rightarrow} \inf A \geq a$, $\sup A \leq b$ ok.

$A := \{\frac{1}{n} : n \in \mathbb{N} \setminus \{0\}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

$\sup A$? $\inf A$?

· $\sup A = \max A = 1$ $\quad (1 \in A$ and $\frac{1}{n} \leq 1$, $\forall n \geq 1)$

· $\inf A = 0$. $x \leq 0 \Rightarrow x \leq a$, $\forall a \in A \Leftrightarrow x \leq \frac{1}{n}$, $\forall n$ $(\frac{1}{n} > 0)$. $\Rightarrow \inf A \geq 0$

$\forall \varepsilon > 0 \Rightarrow \varepsilon$ is not in $m(A)$. In fact $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$. $\frac{1}{n} < \varepsilon$ $[n > \frac{1}{\varepsilon}] \Rightarrow \inf A = 0$.

<u>Remark</u>: Let $A \subseteq \mathbb{R}$ be a set. Then
- A has a maximum $\Leftrightarrow \sup A \in A$.
- A has a minimum $\Leftrightarrow \inf A \in A$.

If it is the case, we say that the sup (or inf) is achieved.

$I = [0, 2)$

- Is the sup achieved? the sup is 2. It is not achieved.
- Is the inf achieved? $\inf I = 0$ is achieved

· <u>Remark</u>: · the sup is never achieved if it is not in $\mathbb{R}$.
$I = (0, \infty) \Rightarrow \sup I = \infty$ is not achieved.
· Let $A \subseteq \mathbb{R}$. $\sup A = -\infty \Leftrightarrow A = \emptyset$
$\inf A = +\infty \Leftrightarrow A = \emptyset$

<u>Powers and roots</u>:

<u>Def</u>: Let $x \in \mathbb{R}$, $n \in \mathbb{Z}$

$$x^n := \begin{cases} \underbrace{x \cdots x}_{n \text{ times}} & n > 0 \\ 0 & n = 0,\ x \neq 0 \\ \frac{1}{x^{-n}} & n < 0,\ x \neq 0. \end{cases}$$

$x = 2$, $n = 2 \Rightarrow x^2 = 2^2 = 4$, $x = -2$, $x^2 = (-2)^2 = 4$

$x = 3$, $n = -2 \Rightarrow x^n = \frac{1}{3^2} = \frac{1}{9}$; $x = -3$, $n = -2 \Rightarrow x^n = \frac{1}{(-3)^2} = \frac{1}{9}$

$x = \frac{1}{2}$, $n = 2 \Rightarrow x^n = (\frac{1}{2})^2 = \frac{1}{4}$

$x = \frac{1}{3}$, $n = -2 \Rightarrow x^n = (\frac{1}{3})^{-2} = \frac{1}{(\frac{1}{3})^2} = \frac{1}{\frac{1}{9}} = 9$

$x = -\frac{1}{2}$, $n = 3 \Rightarrow x^n = (-\frac{1}{2})^3 = \frac{1}{(-2)^3} = -\frac{1}{8}$

· Let $y \geq 0$, $n \in \mathbb{N} \setminus \{0\}$,

<u>Def</u>: $\sqrt[2n]{y} := \sup\{x \in \mathbb{R} : 0 \leq x^{2n} \leq y\}$

· $y = 4$, $n = 2$. $\sqrt{4} = 2$ $\quad (\sqrt[2]{\ } =: \sqrt{\ })$

· <u>Remark</u>: · $\forall x \geq 0$, $n \in \mathbb{N} \setminus \{0\}$, $\sqrt[2n]{x^{2n}} = x$.
· $\forall y \geq 0$, $(\sqrt[2n]{y})^{2n} = y$