Exercises: Sheet 8, 3 b)

$$\lim_{M\to\infty} \int_1^M x^\alpha\, dx = \lim_{M\to\infty} \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_1^M = \qquad \alpha \neq -1$$

$$= \lim_{M\to\infty} \left(\frac{M^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1}\right) = \begin{cases} -\frac{1}{\alpha+1} > 0 & \alpha+1<0 \\ +\infty & \alpha+1>0 \end{cases}$$

Notation: $F(x)\Big]_a^b = F(b) - F(a)$

$\alpha = -1$

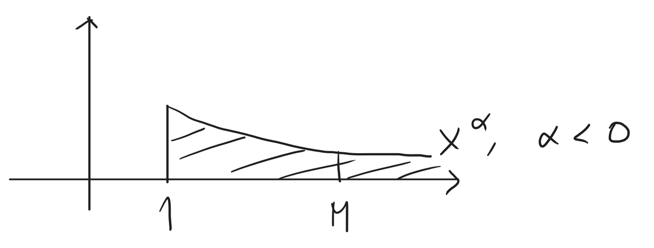
$$\lim_{M\to\infty} \int_1^M x^{-1}\, dx = \lim_{M\to\infty} \log x\Big]_1^M = \lim_{M\to\infty} (\log M - \underbrace{\log 1}_{=0}) = +\infty$$

$$\int x^{-1} dx = \begin{cases} \log x + c & \forall x > 0 \\ \log(-x) + c & \forall x < 0 \end{cases} \qquad \left[\log(-x)' = \frac{1}{-x}(-1) = \frac{1}{x}, \ \forall x<0 \right]$$

To sum up, $\int x^{-1} dx = \log|x| + c$, $\forall x \neq 0$.

Finally:
$$\lim_{M\to\infty} \int_1^M x^\alpha dx = \begin{cases} -\frac{1}{\alpha+1} & \text{if } \alpha < -1 \\ +\infty & \text{if } \alpha \geq -1. \end{cases}$$

geometric interpretation:



$$\left[L\int_1^{+\infty} x^\alpha dx = \begin{cases} -\frac{1}{\alpha+1} & \text{if } \alpha<-1 \\ +\infty & \text{if } \alpha \geq -1 \end{cases} \right.$$ (Lebesgue integral)]
curiosity

Sheet 7:

1, a) Determine the domain and compute the derivative of $f(x) = \tanh^{-1}(x)$.

$sh\, x := \frac{e^x - e^{-x}}{2}$ $\quad D(sh) = \mathbb{R}$; $ch\, x = \frac{e^x + e^{-x}}{2}$ $\quad D(ch) = \mathbb{R}$.

$th\, x = \frac{sh\, x}{ch\, x}$, what is $D(th) = \mathbb{R}$.

$$(sh\,x)' = \frac{(e^x)' - (e^{-x})'}{2} = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = ch\,x$$

$(x^\alpha)' = \alpha x^{\alpha+1}$

$(e^x)' = e^x$; $(e^{-x})' = (e^{-x})(-x)' = -e^{-x}$

$(ch\,x)' = \frac{e^x - e^{-x}}{2} = sh\,x$

$$(th\,x)' = \left(\frac{sh\,x}{ch\,x}\right)' = \frac{(sh\,x)'ch\,x - (ch\,x)'sh\,x}{ch^2 x} = \frac{ch^2 x - sh^2 x}{ch^2 x} = 1 - th^2 x$$

$$\left(\frac{f}{g}\right)' = \left(f\cdot\frac{1}{g}\right)' = f'\cdot\frac{1}{g} + \left(\frac{1}{g}\right)' f = \frac{f'}{g} + f\,(g^{-1})' = \frac{f'}{g} + f\left(\frac{-1}{g^2}\right)g' = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dy}(y^{-1}) = -1\cdot(y^{-2}) = -\frac{1}{y^2}$$

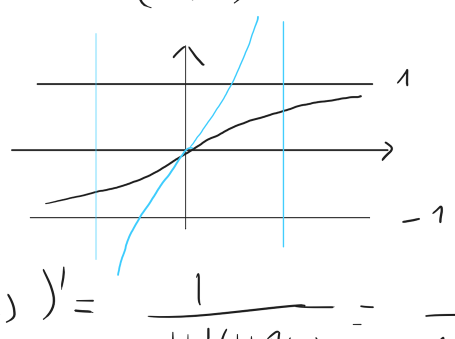
$|th\,x| < 1$, $\forall x \in \mathbb{R}$, since $|sh\,x| < ch\,x$ $\Rightarrow$ $(th\,x)' = 1 - th^2(x) > 0$, $\forall x \in \mathbb{R}$.

$e^x - e^{-x} < e^x + e^{-x}$

$\Rightarrow$ th is invertible. Let us call th^{-1} or arcth the inverse.

what is $D(th^{-1}) = ?$

$D(th^{-1}) = (-1, 1)$



$$(th^{-1}(x))' = \frac{1}{th'(th^{-1}(x))} = \frac{1}{1 - (th(th^{-1}(x)))^2} = \frac{1}{1-x^2}$$

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

$f(f^{-1}(x)) = x$

$th^{-1}(0) = 0$

$$th^{-1}(x) = \int_0^x \frac{1}{1-t^2}\, dt$$

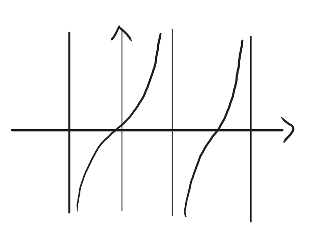
$$(a^x)' = \left(e^{\log a^x}\right)' = \left(e^{x\log a}\right)' = \log a\; e^{x\log a} = (\log a)\, a^x$$

1, b) $f(x) = \tan\left(\frac{1}{2(x^2+1)}\right)$ $\qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$ $\qquad Dom(\tan) = \{\theta\in\mathbb{R} : \cos\theta \neq 0\}$

$= \{\theta \in \mathbb{R} : \theta \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$

$Dom(f) = \mathbb{R}$.

$\frac{1}{2(x^2+1)} \neq \frac{\pi}{2} + k\pi$, $\forall k \in \mathbb{Z}$.



$$0 < \frac{1}{2(x^2+1)} < \frac{1}{2} < \frac{\pi}{2}$$

$x^2+1 > 1$

$$f'(x) = \tan'\left(\frac{1}{2(x^2+1)}\right)\cdot\left(\frac{1}{2(x^2+1)}\right)' = \left(1 - \tan^2\left(\frac{1}{2(x^2+1)}\right)\right)\frac{1}{2}\,\frac{(1)'(x^2+1) - 1\cdot(x^2+1)'}{(x^2+1)^2} =$$

$$= \left(1 - \tan^2\left(\frac{1}{2(x^2+1)}\right)\right)\frac{1}{2}\,\frac{-2x}{(x^2+1)^2} = -\left(1 - \tan^2\left(\frac{1}{2(x^2+1)}\right)\right)\frac{x}{(x^2+1)^2}$$

3, b) $f(x) = x\sin\left(\frac{1}{x}\right)$

$D(f) = \mathbb{R}\setminus\{0\}$

$$f'(x) = (x)'\sin\left(\tfrac{1}{x}\right) + x\left(\sin\tfrac{1}{x}\right)' = \sin\left(\tfrac{1}{x}\right) + x\cos\left(\tfrac{1}{x}\right)\left(\tfrac{1}{x}\right)' = \sin\left(\tfrac{1}{x}\right) + x\cos\left(\tfrac{1}{x}\right)\frac{-1}{x^2} = \sin\left(\tfrac{1}{x}\right) - \tfrac{1}{x}\cos\left(\tfrac{1}{x}\right)$$

3, d) $f(x) = (x^2-1)^{\frac{1}{x-1}}$

$D(f) = \{x \in \mathbb{R} : x - 1 \neq 0 \wedge x^2 - 1 \geq 0\} = (-\infty, -1] \cup (1, +\infty)$.

$x^2 - 1 \geq 0 \iff \mathbb{R}\setminus(-1,1) = (-\infty,-1] \cup [1,\infty)$

$$f'(x) = \left(e^{\log (x^2-1)^{\frac{1}{x-1}}}\right)' = \left(e^{\frac{\log(x^2-1)}{x-1}}\right)' = e^{\frac{\log(x^2-1)}{x-1}}\left(\frac{\log(x^2-1)}{x-1}\right)'$$

$$= (x^2-1)^{\frac{1}{x-1}}\,\frac{(\log(x^2-1))'(x-1) - (x-1)'\log(x^2-1)}{(x-1)^2}$$

$$= \frac{(x^2-1)^{\frac{1}{x-1}}}{(x-1)^2}\left(\frac{1}{x^2-1}(x-1)\cdot 2x - \log(x^2-1)\right) =$$

$$= \frac{(x^2-1)^{\frac{1}{x-1}}}{(x-1)^2}\left(\frac{2x}{x+1} - \log(x^2-1)\right)$$