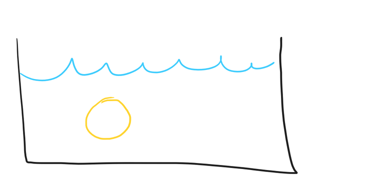
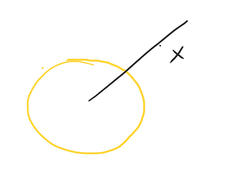
Allen-Cahn equation (dimension $N=1$).

$-u'' = u - u^3 \qquad \mathbb{R}$

we want to find a solution $u:\mathbb{R}\to(-1,1)$ s.t. $u(0)=0$, $u'(t)>0\ \forall t>0$.





$u(x)=$ concentration of oil at x
$u(x)=-1$ means that there is only water at x
$u(x)=1$ means that there is only oil at x.

<u>theorem</u>: $\exists$ a unique solution $u:\mathbb{R}\to(-1,1)$ to the problem

$$\begin{cases} -u'' = u-u^3 & \mathbb{R} \\ u(0)=0, \ \lim_{t\to\pm\infty} u(t)=\pm 1 & \\ u'(t)>0 \quad \forall t>0 & \end{cases}$$

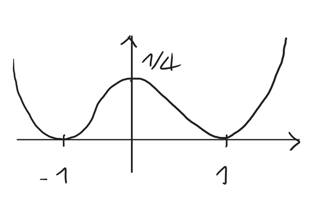
In particular, this solution is given by $u(t)=th(t/\sqrt{2}) = v_*(t)$.

<u>proof</u>: $-u''=u-u^3 \iff -u''u'=(u-u^3)u' \iff -u''u'=-W'(u)u' \iff u''u'=W'(u)u'$

$$\int(u-u^3)\,du = \frac{u^2}{2}-\frac{u^4}{4}+c = \frac{-u^4+2u^2+4c}{4} = -\frac{u^4-2u^2+4c}{4} \underset{c=\frac{1}{4}}{=} -\frac{u^4-2u^2+1}{4} = -\frac{(u^2-1)^2}{4} = -W(u)$$

$\int t^\alpha\,d\alpha = \frac{t^{\alpha+1}}{\alpha+1}+c'$, $\alpha\neq -1$

$W(u) := \frac{(u^2-1)^2}{4} \geq 0$



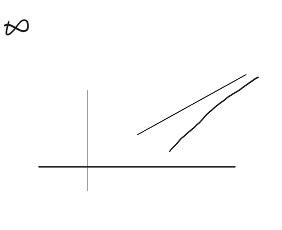
Double-well potential.

$$u''u' = W'(u)u' \iff \left(\frac{(u')^2}{2}\right)' = (W(u))' \iff \underset{\downarrow t\to+\infty\ 0}{\frac{(u')^2}{2}} = \underset{\downarrow t\to+\infty\ 0}{W(u)} + c' \iff 0 = 0 + c \iff c=0.$$

$$\left(\frac{(u')^2}{2}\right)' \underset{1}{=} \frac{1}{2}\left((u')^2\right)' \underset{2}{=} \frac{1}{\cancel{2}}\cancel{2}\,u'(u')' \underset{3}{=} u'u''$$

$[(\alpha f)' = \alpha f'\ ;\ (g^2)' \underset{\text{ch. rule}}{=} 2g\cdot g']$

$\lim_{t\to\pm\infty} u(t)=\pm 1$. $\quad \lim_{t\to\pm\infty} u'(t)=0$. (otherwise u would be unbounded).



if $\lim_{t\to\pm\infty} u(t)=\pm1 \Rightarrow \exists \lim_{t\to\pm\infty} W(u(t)) \Rightarrow \exists \lim_{t\to\pm\infty} u'(t) =: \alpha^{\pm} \geq 0$

if $\alpha^+>0 \Rightarrow u$ is unbounded (the same for α^-).

$$\frac{(u')^2}{2} = W(u) \iff (u')^2 = 2W(u) = 2\frac{(u^2-1)^2}{4} = \frac{(u^2-1)^2}{2}$$

$$\iff u' = \frac{u^2-1}{\sqrt{2}} > 0 \iff \frac{u'}{u^2-1} = \frac{1}{\sqrt{2}} \iff \int_0^t \frac{u'(s)}{1-u^2(s)}\,ds = \int_0^t \frac{ds}{\sqrt{2}} = \frac{1}{\sqrt{2}}s\Big]_0^t = \frac{t}{\sqrt{2}}$$

$$\int_0^t \frac{1}{\sqrt{2}}\,ds = \frac{1}{\sqrt{2}}\int_0^t s^0\,ds = \frac{1}{\sqrt{2}}\cdot\frac{s}{1}\Big]_0^t = \frac{t}{\sqrt{2}} - \frac{0}{\sqrt{2}} = \frac{t}{\sqrt{2}}.$$

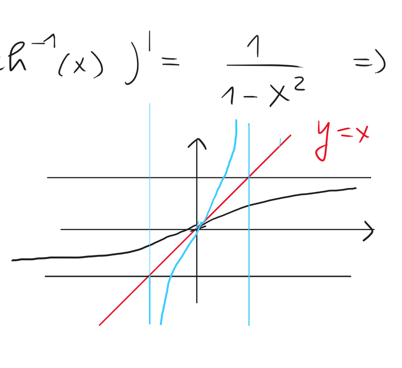
$\int_a^b f(t)\,ds = F(b)-F(a)$

compute the primitive

$$\int_0^t \frac{u'(s)}{1-u^2(s)}\,ds = \frac{t}{\sqrt{2}} \iff th^{-1}(u(s))\Big]_0^t = t/\sqrt{2} \iff th^{-1}(u(t)) - th^{-1}(0) = t/\sqrt{2} \iff th^{-1}(u(t)) = t/\sqrt{2}$$

$\iff u(t) = th(t/\sqrt{2})$

$$(th^{-1}(x))' = \frac{1}{1-x^2} \Rightarrow (th^{-1}(u(s)))' = \frac{1}{1-u^2(s)}u'(s)$$



January 19th, February 3rd, February 18th.

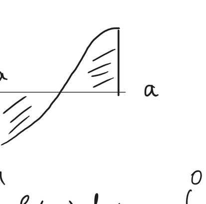
<u>Exercises</u> Sheet 8, 2,a)

$$\int_{-1}^1 \underbrace{\tan\left(\frac{\pi x^3}{2(x^2+1)}\right)\log(x^4+1)}_{f}\,dx = 0$$

because the integrand is odd.
$\|$
the function f.

look at the domain and the symmetries

$f:[-a,a]\to\mathbb{R}$ is odd $\Rightarrow \int_{-a}^a f(x)\,dx = 0$



$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx = -\int_a^0 f(-x')\,dx' + \int_0^a f(x)\,dx = \int_a^0 f(x')\,dx' + \int_0^a f(x)\,dx$$

$x'=-x$
$dx'=-dx$

$$= -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0$$

1,a) $$\int\frac{x+1+2x^2}{\sqrt{x}}\,dx = \int\left(\sqrt{x}+\frac{1}{\sqrt{x}}+2x^{3/2}\right)dx = \int\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}+2x^{3/2}\right)dx =$$

$$= \frac{x^{3/2}}{3/2} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + 2\cdot\frac{x^{5/2}}{5/2} + c = \frac{2}{3}x^{3/2} + 2x^{1/2} + \frac{4}{5}x^{5/2} + c'$$

1,b) $$\int\frac{x+1}{x}\,dx = \int\left(1+\frac{1}{x}\right)dx = x + \log x + c$$

$\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1}+c$ if $\alpha\neq -1$ $\qquad \left(\frac{x^{\alpha+1}}{\alpha+1}\right)' = \frac{\cancel{(\alpha+1)}x^{(\alpha+1)-1}}{\cancel{\alpha+1}} = x^\alpha$

$\int x^{-1}\,dx = \log x + c'$ if $\alpha=-1$ $\qquad (\log x)' = \frac{1}{x} = x^{-1}$

$\int c\,dx = c\int 1\,dx = cx+b$ (linearity of the integral)

1,c) $$\int\frac{\log x}{x}\,dx = -\int\log\left(\frac{1}{y}\right)y\frac{1}{y^2}\,dy = -\int\log\left(\frac{1}{y}\right)\frac{1}{y}\,dy \ \ldots$$

$y=\frac{1}{x}$
$dy = -\frac{1}{x^2}dx \iff dx = -x^2dy = -\frac{1}{y^2}dy$

$\left[y=\log x:\ dy=\frac{dx}{x} \qquad \int\frac{\log x}{x}\,dx = \int y\,dy = \frac{y^2}{2}+c = \frac{(\log x)^2}{2}+c\right]$

$$\int\frac{\log x}{x}\,dx = \int f(x)f'(x)\,dx = \int\left(\frac{f^2(x)}{2}\right)'dx = \frac{f^2(x)}{2}+c = \frac{\log^2 x}{2}+c'$$

where $f(x)=\log x$.

$$\boxed{\int g(f(x))f'(x)\,dx = \int(G(f(x)))'\,dx = G(f(x))+c'}$$

where G is a primitive of g.

$$\int(\sin x)^2\cos x\,dx = \int(1-\cos^2 x)\cos x\,dx = \int\cos x\,dx - \underset{??}{\int(\cos x)^3\,dx}$$

$\int(\sin x)^2\cos x = \int\sin x\frac{\sin 2x}{2}$ $\quad$ stuck.

$$\int(\sin x)^2\cos x\,dx = \int\left(\frac{(\sin x)^3}{3}\right)'dx = \frac{(\sin x)^3}{3}+c'$$

$g(t)=t^2$, $f(x)=\sin x$

$$\int(\underset{y}{\sin x})^2\underbrace{\cos x\,dx}_{dy} = \int y^2\,dy = \frac{y^3}{3}+c = \frac{(\sin x)^3}{3}+c'.$$

$y=\sin x$
$dy=\cos x\,dx$

2,b) $$\int_0^1 x\sqrt{x^2+1}\,dx = \frac{1}{2}\int_0^1 2x\sqrt{x^2+1}\,dx \qquad \int_a^b f(x)\,dx = F(b)-F(a),\ F'=f$$

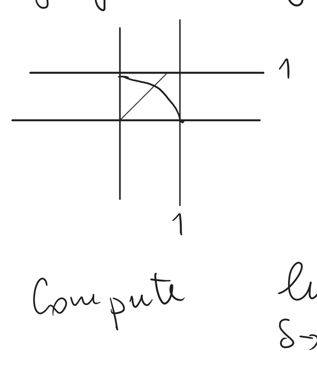
$2x=(x^2+1)'$
$y=x^2+1$
$y'=2x$
$dy=y'(x)dx=2x\,dx$

$$= \frac{1}{2}\int_1^2\sqrt{y}\,dy = \frac{1}{2}\frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big]_1^2 = \frac{1}{2}\frac{y^{3/2}}{3/2}\Big]_1^2 = 3\left[y^{3/2}\right]_1^2 = 3(2^{3/2}-1) = 3(2\sqrt{2}-1)$$

2,c) $$\int_0^1\frac{x}{1+x^4}\,dx = \frac{1}{2}\int_0^1\frac{2x}{1+x^4}\,dx = \frac{1}{2}\int_0^1\frac{2x}{1+(x^2)^2}\,dx = \frac{1}{2}\int_0^1\frac{dy}{1+y^2}$$

$y=x^2$
$dy=y'\,dx=2x\,dy$

$$= \frac{1}{2}\left[\arctan y\right]_0^1 = \frac{1}{2}\left[\frac{\pi}{4}-0\right] = \frac{\pi}{8}$$

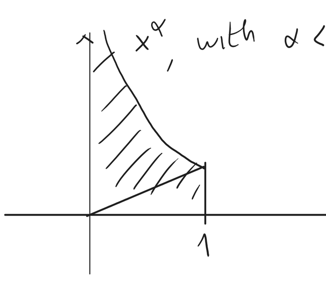


3,a) Compute $\lim_{\delta\to0}\int_\delta^1 x^\alpha\,dx$ for $\alpha\in\mathbb{R}$.

$$\lim_{\delta\to0}\int_\delta^1 x^\alpha\,dx = \lim_{\delta\to0}\left[\frac{x^{\alpha+1}}{\alpha+1}\right]_\delta^1 = \lim_{\delta\to0}\left(\frac{1}{\alpha+1}-\frac{\delta^{\alpha+1}}{\alpha+1}\right) = \begin{cases}\frac{1}{\alpha+1} & \alpha+1>0 \\ +\infty & \alpha+1<0\end{cases} \Bigg|\ \alpha\neq-1$$

case $\alpha=-1$: $$\lim_{\delta\to0}\int_\delta^1 x^{-1}\,dx = \lim_{\delta\to0}\left[\log x\right]_\delta^1 = \lim_{\delta\to0}(-\log\delta) = +\infty$$

In the end, we have $$\lim_{\delta\to0}\int_\delta^1 x^\alpha\,dx = \begin{cases}\frac{1}{\alpha+1} & \text{if } \alpha>-1 \\ +\infty & \text{if } \alpha\leq-1.\end{cases}$$



[the function x^α is Lebesgue integrable in $(0,1)$ for $\alpha>-1$, while it is not for $\alpha\leq-1$.]

curiosity.