Exercises: Sheet 5

1,a) Determine the complement of the set $A = (-\infty, 0) \cup (1, \infty)$ in $\mathbb{R}$.

$A^c = \mathbb{R} \setminus A = [0, 1]$

1,b) $B \cap C = \emptyset$

2, a) $e^{1-|x+3|} > 1 \iff 1 - |x+3| > \log 1 = 0 \iff -|x+3| > -1 \iff |x+3| < 1$

$\iff \begin{cases} x \geq -3 \\ x+3 < 1 \end{cases} \vee \begin{cases} x < -3 \\ x+3 > -1 \end{cases} \iff \begin{cases} x \geq -3 \\ x < -2 \end{cases} \vee \begin{cases} x < -3 \\ x > -4 \end{cases} \iff -3 \leq x < -2 \;\vee\; -4 < x < -3$

$x \in [-3, -2) \;\vee\; x \in (-4, -3)$

$\iff x \in (-4, -2)$

alternative: $|x+3| < 1 \iff -1 < x+3 < 1 \iff -4 < x < -2$

2,b) $x^6 + 3x^3 + 2 > 0$. $a = x^3 \in \mathbb{R}$, $a^2 + 3a + 2 > 0 \iff a \in (-\infty, -2) \cup (-1, +\infty) \iff a < -2 \vee a > -1$

$\iff x^3 < -2 \vee x^3 > -1 \iff \boxed{x < \sqrt[3]{-2} \vee x > -1}$

$a_\pm = \frac{-3+1}{2} / \frac{-3-1}{2} = -1 / -2$

$\Delta = 9 - 8 = 1$

$a^2 + 3a + 2 = 0 \iff a = -1 \vee a = -2.$

$a^2 + 3a + 2 = (a+1)(a+2) > 0$

		-2		-1	
$a+1$	$-$		$-$		$+$
$a+2$	$-$		$+$		$+$
	$+$		$-$		$+$

. $(x^2 < -2 \vee x^2 > -1$. the solution is $\forall x \in \mathbb{R})$

2,c) $(\log_3 x)^2 + 3 \log_3 x + 2 > 0$. $y = \log_3 x$

$y^2 + 3y + 2 > 0 \iff y < -2 \vee y > -1 \iff \log_3 x < -2 \vee \log_3 x > -1 \iff \begin{cases} x < \frac{1}{9} \\ x > 0 \end{cases} \vee x > \frac{1}{3}$

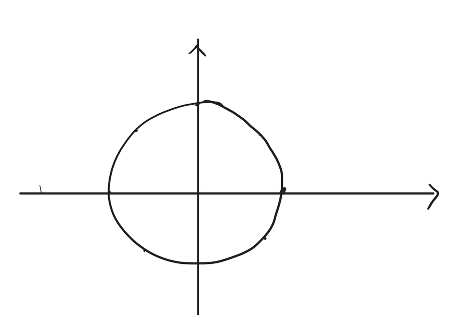
$\iff \boxed{0 < x < \frac{1}{9} \vee x > \frac{1}{3}}$

2,d) $\cos^2(4x) - \sin^2(4x) = 1 \iff \cos(2 \cdot 4x) = 1 \iff \cos(8x) = 1 \iff 8x = 2\pi k, k \in \mathbb{Z} \iff x = \frac{\pi k}{4}, k \in \mathbb{Z}$

$[\cos^2\theta - \sin^2\theta = \cos 2\theta]$

$8x = \theta$

$\cos\theta = 1 \iff \theta = 0 + 2\pi k = 2\pi k, \; k \in \mathbb{Z}$



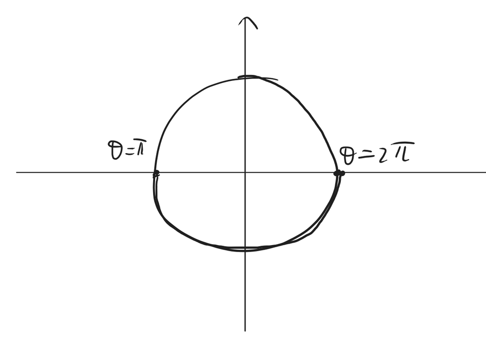
$\cos(8x) = 1 \iff 8x \in \{2\pi k : k \in \mathbb{Z}\} \iff x \in \{\frac{\pi k}{4} : k \in \mathbb{Z}\}$

2,e) $e^{2\sin x \cos x} < 1 \iff 2 \sin x \cos x < 0 = \log 1 \iff \sin(2x) < 0 \iff 2x \in \bigcup_{k \in \mathbb{Z}} (\pi + 2\pi k, 2\pi + 2\pi k)$

$2\sin x \cos x = \sin(2x)$

$2x = \theta$

$\sin\theta < 0 \iff \theta \in \bigcup_{k \in \mathbb{Z}} (\pi + 2\pi k, 2\pi + 2\pi k)$



$\iff x \in \bigcup_{k \in \mathbb{Z}} \left(\frac{\pi}{2} + \pi k, \pi(1+k)\right)$

example: $\sin x = 0 \iff x = 0 + 2\pi k \vee x = \pi + 2\pi k, k \in \mathbb{Z} \iff x \in \bigcup_{k \in \mathbb{Z}} \{k\pi\}$

$\iff x = k\pi$, for some $k \in \mathbb{Z}$.

3) $f(x) = \log_3\left(\frac{x^2+1}{x^2-1}\right)$

a) Determine the domain $D(f)$.

$\frac{x^2+1}{x^2-1} > 0$:

. $x \neq \pm 1$, $x^2 + 1 > 0 \;\; \forall x \in \mathbb{R}$

$x^2 - 1 > 0 \iff (x-1)(x+1) > 0 \iff x \in (-\infty, -1) \cup (1, +\infty)$

		-1		1	
$x-1$	$-$		$-$		$+$
$x+1$	$-$		$+$		$+$
	$+$		$-$		$+$

$D(f) = (-\infty, -1) \cup (1, +\infty)$.

b) Determine the zeros of f.

$\log_3\left(\frac{x^2+1}{x^2-1}\right) = 0 \iff \frac{x^2+1}{x^2-1} = 1$ impossible $\left[\begin{array}{c} x^2 + 1 = x^2 - 1 \\ 1 = -1 \text{ false.} \end{array}\right]$

c) if $f: D(f) \to \mathbb{R}$ even? $f(-x) = \log_3\left(\frac{(-x)^2+1}{(-x)^2-1}\right) = \log_3\left(\frac{x^2+1}{x^2-1}\right) = f(x)$.

$f(x) = f(-x) \;\; \forall x \in D(f)$.

d) Is $f: D(f) \to \mathbb{R}$ injective? No, because it is even, so $f(x) = f(-x) \;\; \forall x \in D(f)$.

e) Determine the set $f^{-1}((1, +\infty))$.

$f^{-1}(I) = \{x \in D(f) : f(x) \in I\} = \{x \in D(f) : f(x) \in (1, +\infty)\} = \{x \in D(f) : f(x) > 1\}$

$\uparrow$

$I = (1, +\infty)$

$f: X \to Y$, $Z \subset Y$, $f^{-1}(Z) = \{x \in X : f(x) \in Z\}$ $\quad [X = D(f)]$

$\Big[f^{-1}((-\infty, 10)) = \{x \in D(f) : f(x) \in (-\infty, 10)\} = \{x \in D(f) : f(x) < 10\} \Big]$

$f(x) = y; \;\; y \in (-\infty, 1) \iff y < 1 \iff f(x) < 1$

$f(x) = \log_3\left(\frac{x^2+1}{x^2-1}\right) > 1 \iff 3^{\log_3\left(\frac{x^2+1}{x^2-1}\right)} > 3^1 \iff \frac{x^2+1}{x^2-1} > 3 \iff \frac{x^2+1-3(x^2-1)}{x^2-1} > 0$

$\iff \frac{x^2+1-3x^2+3}{x^2-1} > 0 \iff \frac{4-2x^2}{x^2-1} > 0$, $x \in D(f) = (-\infty, -1) \cup (1, +\infty)$

$x^2 - 1 > 0, \forall x \in D(f)$, so it is enough to study $\begin{cases} 4 - 2x^2 > 0 \\ x \in (-\infty, -1) \cup (1, +\infty) \end{cases}$

$\iff \begin{cases} 2x^2 < 4 \\ x \in (-\infty, -1) \cup (1, \infty) \end{cases} \iff \begin{cases} x^2 < 2 \\ x \in D(f) \end{cases} \iff \begin{cases} -\sqrt{2} < x < \sqrt{2} \\ x < -1 \vee x > 1 \end{cases} \iff \begin{cases} x \in (-\sqrt{2}, \sqrt{2}) \\ x \in (-\infty, -1) \cup (1, \infty) \end{cases}$

$\iff \boxed{(-\sqrt{2}, -1) \cup (1, \sqrt{2}) = f^{-1}((1, +\infty))}$

4) $f(x) = (\log(x))^3 - 1$

a) Determine the domain $D(f)$. $\boxed{D(f) = (0, \infty)}$

b) Say if $f: D(f) \to f(D(f))$ is invertible and, if it is so, determine the image $f(D(f))$.

$f(x) = (\log x)^3 - 1 = y \iff (\log x)^3 = y + 1 \iff \log x = \sqrt[3]{y+1} \iff x = e^{\sqrt[3]{y+1}}$, $\forall y \in \mathbb{R}$.

$\forall y \in \mathbb{R}$, we can solve equation $f(x) = y \Rightarrow f$ is invertible, $f(D(f)) = \mathbb{R}$

and $f^{-1}(y) = e^{\sqrt[3]{y+1}}$, $\forall y \in \mathbb{R}$. $[D(f^{-1}) = f(D(f))]$