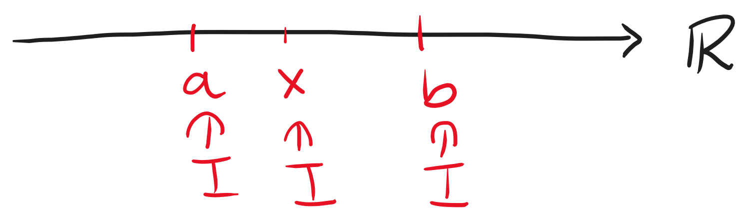
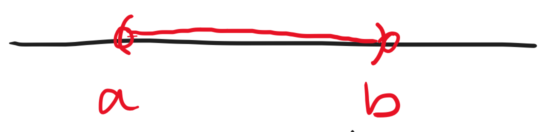
The extrema of a set (section 3.4)

Def: A set $I \subseteq \mathbb{R}$ is said to be an interval if $\forall a,b \in I$, $a<b$ and $\forall x \in \mathbb{R}$, with $a<x<b$, we have $x \in I$



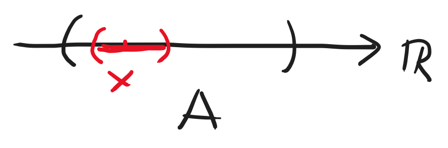
"A set is an interval if there are no gaps between its points"

Example: $I=(a,b)=\{x \in \mathbb{R} : a<x<b\}$, $a,b \in \mathbb{R}$, $a<b$.



Definition Let $A \subseteq \mathbb{R}$ be a set. Then we say that

(i) A is open if $\forall x \in A$, $\exists \delta>0 : (x-\delta, x+\delta) \subseteq A$



(ii) A is closed if $\mathbb{R} \setminus A$ is open

(iii) A is bounded if $\exists a,b \in \mathbb{R}$, $a<b$: $A \subseteq (a,b)$.

(iv) A is unbounded if it is not bounded.

(v) the empty set ϕ is open

Definition Let $A \subseteq \mathbb{R}$ be set.

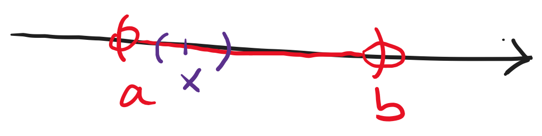
(i) we say that a point $x \in A$ is an interior point if $\exists \delta>0 : (x-\delta, x+\delta) \subseteq A$.

(ii) The set of interior points of A is known as the interior of A and denoted by $\mathring{A}$

Remarks:
- $\mathring{A} \subseteq A$
- $\mathring{A} = A \iff A$ is open.

Examples:

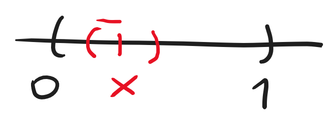
the interval $A=(a,b)$ is open and bounded.

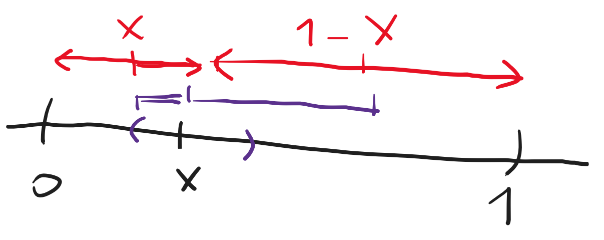


$(x-\delta, x+\delta) \subseteq A$

$\delta = \dfrac{x-a}{2}$ $\quad [x-a<b-x]$

$I=(0,1)$ is an open bounded interval





$\delta := \min\left\{ \frac{x}{2}, \frac{1-x}{2} \right\} > 0$, $x<\frac{1}{2}$

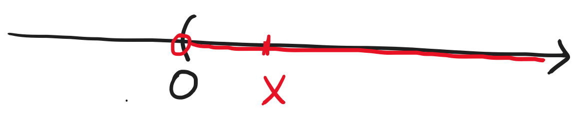
$(x-\delta, x+\delta) \subseteq (0,1)$

if $x-\delta < y < x+\delta \Rightarrow 0<y<1$

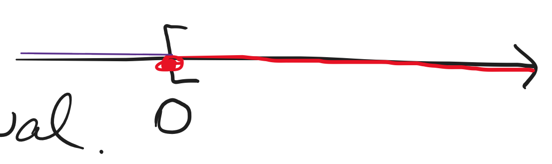
if $y > x-\delta \Rightarrow y > x - \frac{x}{2} = \frac{x}{2} > 0$

if $y < x+\delta \Rightarrow y < x + \frac{x}{2} = \frac{3}{2}x < 1 \iff x < \frac{2}{3}$

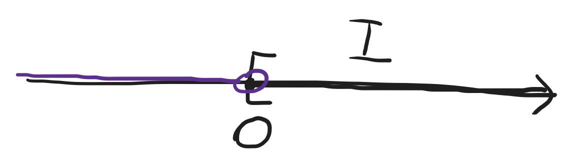
- $I=(0,+\infty)=\{x \in \mathbb{R} : x>0\}$ is an open, unbounded interval.



- $I=[0,+\infty)=\{x \in \mathbb{R} : x \geq 0\}$ is a closed, unbounded interval.



$I^c = \mathbb{R} \setminus I = (-\infty, 0) = \{x \in \mathbb{R} : x<0\}$

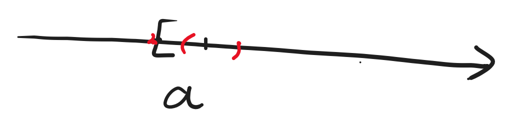


$\neg(x \geq 0) \iff x<0$

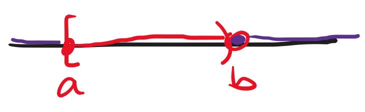
$I^c = \{x \in \mathbb{R} : x \notin I\} = \{x \in \mathbb{R} : \neg(x \geq 0)\} = \{x \in \mathbb{R} : x<0\}$

I^c is open $\Longrightarrow I$ is closed

- $I=[0,+\infty)=\{x \in \mathbb{R} : x \geq 0\}$ is a closed, unbounded interval
- $I=[a,+\infty)=\{x \in \mathbb{R} : x \geq a\}$ is a closed, unbounded interval, $\forall a \in \mathbb{R}$

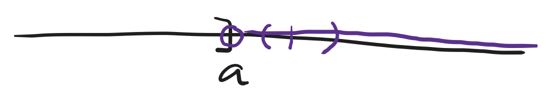


- $I=(-\infty, a)$, $a \in \mathbb{R}$. This is an open, unbounded interval $\forall a \in \mathbb{R}$
- $I=(-\infty, a]$, $a \in \mathbb{R}$. This is a closed, unbounded interval, $\forall a \in \mathbb{R}$
- $\mathbb{R}=(-\infty,\infty)$ is an open, unbounded interval.
- $a,b \in \mathbb{R}$, $a<b$, $I:=[a,b)=\{x \in \mathbb{R} : a \leq x < b\}$



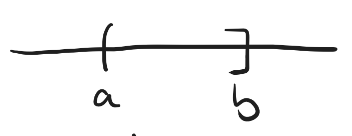
$(I^c = (-\infty, a) \cup [b,\infty))$ I is a bounded interval and its neither closed nor open.

If $I=(-\infty, a] \Rightarrow I^c = (a,+\infty)$ is open



- $a,b \in \mathbb{R}$, $a<b$

$I := (a,b] = \{x \in \mathbb{R} : a<x \leq b\}$



is a bounded interval and it is neither open nor closed.