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Numeric sets

$\mathbb{N} := \{0, 1, 2, \dots\}$ natural numbers.

$\forall n, m \in \mathbb{N}, m+n \in \mathbb{N}, n \cdot m \in \mathbb{N}$.

$n, m \in \mathbb{N} \not\Rightarrow n-m \in \mathbb{N}$. $n=2, m=4$.

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots, \pm n, \dots\}$ integer numbers

$\forall n, m \in \mathbb{Z}, n+m \in \mathbb{Z}, n \cdot m \in \mathbb{Z}, n-m \in \mathbb{Z}$.

$n, m \in \mathbb{Z} \not\Rightarrow \frac{m}{n} \in \mathbb{Z}$; $\frac{1}{2} \in \mathbb{Z}$

$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, n \neq 0\}$ rational numbers

the four operations are well defined.

$\forall p, q \in \mathbb{Q}, p+q \in \mathbb{Q}, p \cdot q \in \mathbb{Q}, p-q \in \mathbb{Q}$. If $q \neq 0, \frac{p}{q} \in \mathbb{Q}$.

$p = \frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{N} \setminus \{0\}, q = \frac{c}{d}, c \in \mathbb{Z}, d \in \mathbb{N}, c \neq 0, d \neq 0$

$\Rightarrow \frac{p}{q} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \in \mathbb{Q}$

Example:

$$\frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

Remark: in the definition of $\mathbb{Q}$ we can also write

$$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}\} \quad (- \cdot - = +)$$

up to now $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$.

Real numbers

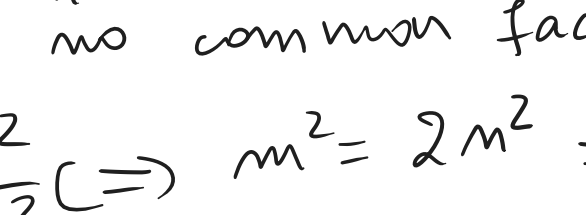
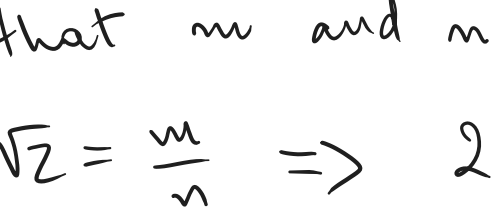
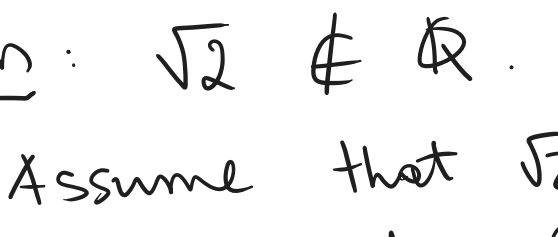
Why do we need to introduce a larger set?

Geometry:

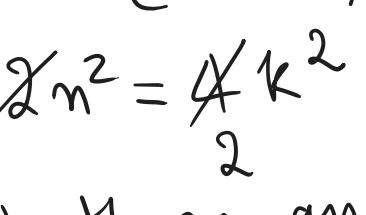
Theorem (Pythagora theorem) let a, b, c be the edges of a rectangle triangle. Then $a^2 + b^2 = c^2$

a and b are the catheti

c is the ipotenuse



$$a=b=1$$



$$c^2 = a^2 + b^2 = 1 + 1 = 2$$

$$\Leftrightarrow c = \sqrt{2}$$

Theorem: $\sqrt{2} \notin \mathbb{Q}$.

proof: Assume that $\sqrt{2} = \frac{m}{n}$, $m, n \in \mathbb{N} \setminus \{0\}$. We can assume that m and n have no common factors.

$$\sqrt{2} = \frac{m}{n} \Rightarrow 2 = \frac{m^2}{n^2} \Leftrightarrow m^2 = 2n^2 \Rightarrow m^2 \text{ is even}$$

$$\Rightarrow m^2 = (2k)^2, \text{ for some } k \in \mathbb{N} \Leftrightarrow m^2 = 4k^2$$

$$\Rightarrow 2k^2 = k^2 \Leftrightarrow m^2 = 2k^2 \Rightarrow m \text{ is even}$$

$$\Rightarrow \text{both } m \text{ and } n \text{ are even. contradiction}$$

(This means that m and n have a common factor that is 2)

we need to introduce a larger set.

How can we define such a larger set?

Thm: Any rational number $q \in \mathbb{Q}$ has a decimal expression which is either finite or periodic.

$$1:2 = 0,5 \quad 1:3 = 0,33\dots = 0,\overline{3}$$

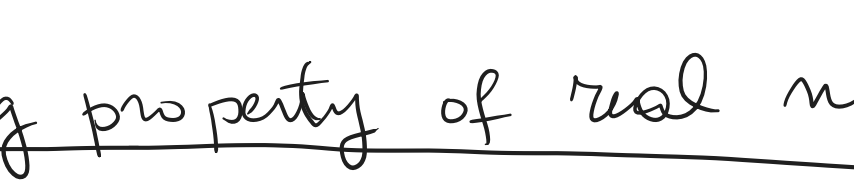
$$\mathbb{R} := \{n, a_1 a_2 \dots a_k \dots, n \in \mathbb{Z}, a_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\}$$

$$1,23678100177\dots \in \mathbb{R}.$$

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

Theorem: (Continuity) $\forall a, b \in \mathbb{R}, \text{ with } a < b, \exists c \in \mathbb{R} : a < c < b$

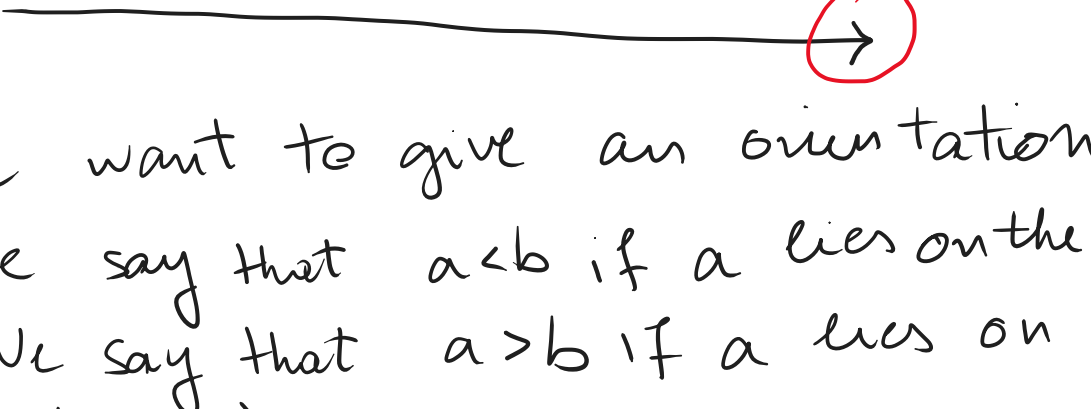
"Continuity means that real numbers can be represented in a straight line"



$$a < c = \frac{a+b}{2} < b$$

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)

$$\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists q \in \mathbb{Q} : x - \varepsilon < q < x + \varepsilon$$



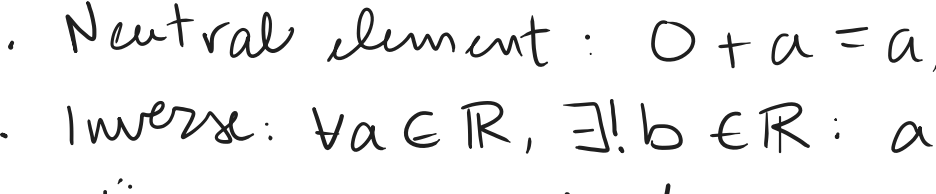
Equivalent form of stating the density theorem is the following:

$$\forall a, b \in \mathbb{R}, \exists q \in \mathbb{Q} : a < q < b$$



Ordering property of real numbers

"the real numbers are a straight line"



We want to give an orientation to $\mathbb{R}$.

We say that $a < b$ if a lies on the left of b .

We say that $a > b$ if a lies on the right of b .

($a, b \in \mathbb{R}$)

We say that $a \geq b$ if $a > b \vee a = b$.

We say that $a \leq b$ if $a = b \vee a < b$.

Some basic properties:

properties of the sum:

Commutative: $\forall a, b \in \mathbb{R}, a+b = b+a$

Associative: $\forall a, b, c \in \mathbb{R}, (a+b) + c = a + (b+c)$

Neutral element: $0 + a = a, \forall a \in \mathbb{R}$ (0 is the neutral element)

Inverse: $\forall a \in \mathbb{R}, \exists ! b \in \mathbb{R} : a+b = 0$, and $b = -a$.

properties of the product:

Commutative: $\forall a, b \in \mathbb{R}, a \cdot b = b \cdot a$

Associative: $\forall a, b, c \in \mathbb{R}, (a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Neutral element: 1 is the neutral element: $\forall a \in \mathbb{R}, a \cdot 1 = a$

Inverse: $\forall a \in \mathbb{R} \setminus \{0\}, \exists ! b \in \mathbb{R} : a \cdot b = 1$, and $b = \frac{1}{a} =: a^{-1}$.

Example: if $a=2$, what is a^{-1} ? $a^{-1} = \frac{1}{2}$, because $2 \cdot \frac{1}{2} = 1$

if $a = \frac{3}{2}$, what is a^{-1} ? $a^{-1} = \frac{2}{3}$, because $\frac{3}{2} \cdot \frac{2}{3} = 1$

If $q = \frac{m}{n} \in \mathbb{Q}, m \neq 0 (\Leftrightarrow q \neq 0)$, then $q^{-1} = \frac{n}{m}$

Because $\frac{m}{n} \cdot \frac{n}{m} = 1$

if $m \in \mathbb{Z}, m \neq 0 \Rightarrow m^{-1} = \frac{1}{m}$

Remark: Integers are fractions with denominator 1.

Properties:

cancellation properties:

① $\forall a, b, c \in \mathbb{R}, a+c = b+c (\Leftrightarrow) a=b$ (sum)

② $\forall a, b, c \in \mathbb{R}, c \neq 0, ac = bc (\Leftrightarrow) a=b$ (product)

proof: ① $a+c = b+c (\Leftrightarrow) a+c-c = b+c-c (\Leftrightarrow) a+0 = b+0 (\Leftrightarrow) a=b$

② $ac = bc (\Leftrightarrow) a \cdot \frac{c}{c} = b \cdot \frac{c}{c} (\Leftrightarrow) a \cdot 1 = b \cdot 1 (\Leftrightarrow) a=b$

Equations (first degree equations)

$a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$. We want to solve equation

$$ax + b = 0.$$

In other words, we want to find $x \in \mathbb{R}$ such that equality is true.

$$ax + b = 0 (\Leftrightarrow) ax + b - b = -b (\Leftrightarrow) ax = -b (\Leftrightarrow) \frac{a}{1} x = -b a^{-1}$$

$$\Leftrightarrow x = -b a^{-1} = -\frac{b}{a}$$

Examples: $2x + 3 = 0 (\Leftrightarrow) 2x = -3 (\Leftrightarrow) x = -\frac{3}{2}$

$$x - 4 = 0 (\Leftrightarrow) x = 4$$

$$2x - 6 = 0 (\Leftrightarrow) x = 3$$

If $a=0$, equation $ax+b=0$ has $\begin{cases} \text{no solution if } b \neq 0 \\ \text{any } x \in \mathbb{R} \text{ is a solution if } b=0 \end{cases}$

Properties of ordering:

① $\forall a, b, c \in \mathbb{R}, a+c > b+c (\Leftrightarrow) a > b$

$a+c < b+c (\Leftrightarrow) a < b$

② $\forall a, b, c \in \mathbb{R}, c > 0, ac > bc (\Leftrightarrow) a > b$

$c < 0, ac > bc (\Leftrightarrow) a < b$

Remark: $\forall a \in \mathbb{R}, 0a = 0$

Example: $a > 0, b \in \mathbb{R}$. We want to "solve" $ax + b > 0$.

In other words, we want to find the values of the unknown $x \in \mathbb{R}$ such that inequality $ax + b > 0$ is satisfied.

$$ax + b > 0 (\Leftrightarrow) ax + b - b > -b (\Leftrightarrow) ax > -b (\Leftrightarrow) \frac{a}{1} x > -b a^{-1}$$

$$\Leftrightarrow x > -\frac{b}{a}$$

$a < 0$, the inequality $ax + b > 0$

$$\Leftrightarrow ax > -b (\Leftrightarrow) x < -\frac{b}{a}$$

$$2x + 3 > 0 (\Leftrightarrow) 2x > -3 (\Leftrightarrow) x > -\frac{3}{2}$$

$$\bullet -2x + 3 > 0 (\Leftrightarrow) 2x - 3 < 0 (\Leftrightarrow) 2x < 3 (\Leftrightarrow) x < \frac{3}{2}$$

$$\bullet -5x + 2 < 0 (\Leftrightarrow) 5x - 2 > 0 (\Leftrightarrow) 5x > 2 (\Leftrightarrow) x > \frac{2}{5}$$

$$\bullet 3x > 2x + 5 (\Leftrightarrow) 3x - 2x > 5 (\Leftrightarrow) x > 5$$

$$ax + b > 0 : x - 5 > 0 (\Leftrightarrow) x > 5$$

$$ax > b \text{ or } ax < b, \text{ or } ax = b$$

$$x + 2 < 3x + 1 (\Leftrightarrow) x - 3x < 1 - 2 (\Leftrightarrow) -2x < -1$$

$$\Leftrightarrow 2x > 1 (\Leftrightarrow) x > \frac{1}{2}$$

Step I: you put all the terms with x on the left and the terms without x on the right

Step II: calculations (sum)

Step III: reduce your selves to $ax \gtrless b$ with $a > 0$

(Multiply by -1 if necessary)

Step IV: divide by $a > 0$.

$$x + 3 \geq 2x + 1$$

$$x - 2x \geq -3 + 1$$

$$-x \geq -2$$

$$x \leq 2$$

$$-2x + 4 > -2x + 3$$

$$-2x + 2x > 3 - 4$$

$$0 \cdot x > -1 (\Leftrightarrow) 0 > -1$$