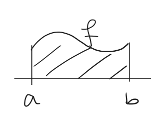
Integrals:

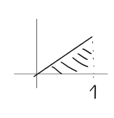


$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Let $f:[a,b]\to\mathbb{R}$ be a function. We say that $F:[a,b]\to\mathbb{R}$ is a primitive of f if $F'=f$. If it is the case, we write $F(x)=\int f(x)dx$.

Theorem (Fundamental theorem of integral calculus)
Let $f:[a,b]\to\mathbb{R}$ be a continuous function, with $a,b\in\mathbb{R}, a<b$. Let $F:[a,b]\to\mathbb{R}$ be a primitive of f. Then $\int_a^b f(x)dx = F(b)-F(a)$.

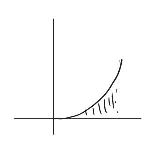
Example: $f(x)=x$



Area $= \frac{1}{2}$

$\int_0^1 x\,dx \overset{?}{=} \frac{1}{2}$: $\int_0^1 x\,dx = \frac{x^2}{2}\Big]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$

$f(x) = x^3$



$$\int_0^1 x^3dx = \frac{x^4}{4}\Big]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$$

proof: f is cont in $[a,b] \Rightarrow f$ is R-integrable. R = Riemann

$\forall \varepsilon>0,\ \exists P_1 := \{a=x_0,\dots,x_n=b\}$: $\int_a^b f(x)dx > U(f,P_1)-\varepsilon$.

$\exists P_2 := \{a=y_0\dots y_m=b\}$: $\int_a^b f(x)dx < L(f,P_2)+\varepsilon$.

$P:=P_1\cup P_2 = \{a=z_0,\dots z_k=b\}$

$$\int_a^b f(x)dx - \varepsilon < L(f,P_2) \le L(f,P) \le U(f,P) \le U(f,P_1) < \int_a^b f(x)dx + \varepsilon$$

$$F(b)-F(a) = \sum_{i=1}^k \left(F(z_i) - F(z_{i-1})\right) = F(b) - F(z_{k-1}) + F(z_{k-1}) - \dots - F(a)$$

$$\underset{\text{Lagrange}}{=} \sum_{i=1}^k f(\xi_i)(z_i - z_{i-1})$$

$\inf_{x\in[z_{i-1},z_i]} f(x) \le f(\xi_i) \le \sup_{x\in[z_{i-1},z_i]} f(x)$ (def of inf and sup)

$$\Rightarrow \underbrace{\sum_{i=1}^k \inf_{x\in[z_{i-1},z_i]} f(x)(z_i-z_{i-1})}_{L(f,P)} \le \sum_{i=1}^k f(\xi_i)(z_i-z_{i-1}) \le \underbrace{\sum_{i=1}^k \sup_{x\in[z_{i-1},z_i]} f(x)(z_i-z_{i-1})}_{U(f,P)}$$

$$\int_a^b f(x)dx - \varepsilon \le L(f,P) \le F(b)-F(a) \le U(f,P) \le \int_a^b f(x)dx + \varepsilon \qquad \square$$

Some useful results:

Integration by parts

Theorem: Let $f,g:[a,b]\to\mathbb{R}$ be differentiable functions.

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)dx$$

$$\int_1^2 \log x\,dx = \int_1^2 \underset{f}{\log x}\cdot \underset{g'}{1}\,dx = \underset{g}{x}\,\underset{f}{\log x}\Big]_1^2 - \int_1^2 \underset{f'}{\frac{1}{x}}\cdot x\,dx = 2\log 2 - \overbrace{1\cdot\log 1}^{0} - \int_1^2 1\,dx$$

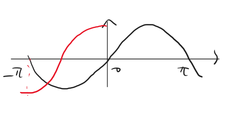
$$= 2\log 2 - x\Big]_1^2 = 2\log 2 - (2-1) = 2\log 2 - 1.$$

Theorem (Change of variables)
Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Let $g:[c,d]\to[a,b]$ be a differentiable function. Then $\int_c^d f(g(x))\,g'(x)\,dx = \int_a^b f(y)dy$

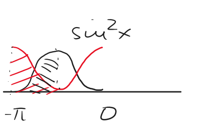
Example:

$$\int_{-1}^1 \sqrt{1-y^2}\,dy = \int_{-\pi}^0 \sqrt{1-\cos^2 x}\,(-\sin x)\,dx = \int_{-\pi}^0 \sqrt{\sin^2 x}\,(-\sin x)dx = \int_{-\pi}^0 (-\sin x)^2 dx = \int_{-\pi}^0 \sin^2 x\,dx$$

$y=\cos x$, $dy = -\sin x\,dx$



$$\boxed{\int_{-\pi}^0 \sin^2 x\,dx = \int_{-\pi}^0 \cos^2 x\,dx}$$ symmetry

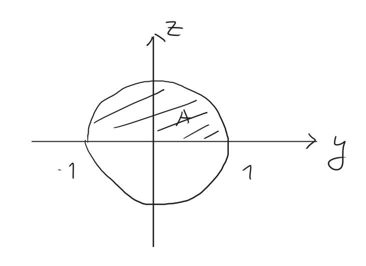


$$\int_{-\pi}^0 \sin^2 dx + \int_{-\pi}^0 \cos^2 x\,dx = \int_{-\pi}^0 1\,dx = \pi = [x]_{-\pi}^0$$

$$\Rightarrow \int_{-\pi}^0 \sin^2 x\,dx = \int_{-\pi}^0 \cos^2 dx = \frac{\pi}{2} \qquad \left[\begin{matrix} a+b=\pi \\ a=b\end{matrix} \Leftrightarrow \begin{matrix} 2a=\pi \\ a=b\end{matrix} \Leftrightarrow a=b=\frac{\pi}{2}\right]$$

$$\int_{-1}^1 \sqrt{1-y^2}\,dy = \frac{\pi}{2}$$

geometric interpretation:



$f(y) = \sqrt{1-y^2} = z \ge 0$

$D(f) = \{y\in\mathbb{R}: 1-y^2\ge 0\} = [-1,1]$

$1-y^2 = (1-y)(1+y)$

		-1	1	
$1-y$	+	+	−
$1+y$	−	+	+
	−	+	−

$1-y^2 = z^2 \Leftrightarrow y^2+z^2=1$; The graph of f is the upper part of the circle of radius 1. $\Rightarrow A = \int_{-1}^1 \sqrt{1-y^2}dy$; but the area of a disc of radius R is $\pi R^2 \Rightarrow$ the area of the unit disc is $\pi \Rightarrow A = \frac{1}{2}\pi = \frac{\pi}{2}$

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

$\int_0^{-1} f(x)dx = -\int_{-1}^0 f(x)dx$.

Differential equations:

Differential equations are equations in which the unknowns is a function and some derivative appear.

Example: (linear equations of degree 1).

$a(t)x' + b(t)x = 0$.

Geology: We want to establish the age of a rock.
Let $x(t)$ be the ratio $= \frac{\text{unstable isotopes}}{\text{total amount of el}}$ at time t.

$x(0)=1$, $x(+\infty)=0$.

problem: We want to compute a law $x(t)$ which, $\forall t>0$, gives the ratio x at time t.

Idea: $x(t)$ is proportional to the decreasing velocity $v(t)$.
$v(t) = -\lambda x(t)$, $\lambda>0$.
$v(t) = x'(t)$

we need to solve the differential equation $\begin{cases} x'(t) = -\lambda x(t) \\ x(0)=1\end{cases}$

Cauchy problem

$\begin{cases} x'(t) = -\lambda x(t) \\ x>0\end{cases}$ $\quad \int \frac{x'(t)}{x(t)}dt = -\int \lambda dt \Leftrightarrow \log x(t) = -\lambda t + c' \Leftrightarrow x(t) = e^{c-\lambda t}$

$\Leftrightarrow \boxed{x(t) = \tilde{c}\,e^{-\lambda t}}$

$\frac{d}{dt}\left(\log(x(t))\right) = \frac{1}{x(t)}x'(t) = \frac{x'(t)}{x(t)}$; $(-\lambda t)' = -\lambda(t)' = -\lambda$

$t \overset{x}{\longmapsto} x(t) =: y \overset{\log}{\longmapsto} \log(y)$

$f'=g' \Rightarrow f-g = c\in\mathbb{R}$

$f=x,\ g=x+1$

$\begin{cases} x'(t) = -\lambda x(t) \\ x>0\end{cases} \Rightarrow \exists\, c>0:\ x(t) = c\,e^{-\lambda t}$ $\quad [(ce^{-\lambda t})' = c(e^{-\lambda t})' = c(-\lambda)e^{-\lambda t} = -\lambda x(t)]$

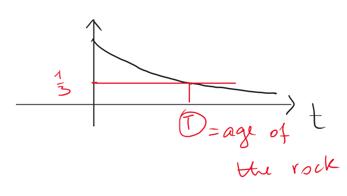
$(t^\alpha)' = \cancel{\alpha t^{\alpha-1}}$; $(e^{\alpha t})' = e^{\alpha t}\cdot\alpha$ ✓

$t \mapsto \alpha t \mapsto e^{\alpha t}$

$x(0)=1 \Leftrightarrow 1 = c\,e^{-\lambda\cdot 0} = c \Leftrightarrow c=1$

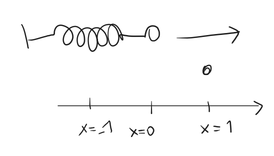
the solution to the Cauchy problem $\begin{cases} x' = -\lambda x \\ x(0)=1\end{cases}$ is unique. In particular it is

$x(t) = e^{-\lambda t}$ $[c=1]$



$x(+\infty)=0$, $x' = -\underset{>0}{\lambda}\,\underset{>0}{x} < 0$

$y = e^{-\lambda t} \Leftrightarrow \log y = -\lambda t \Leftrightarrow t = -\frac{1}{\lambda}\log(y)$



$\vec{F} = m\,\vec{a}$ (Newton Law)

$\vec{F} = -\lambda\vec{x} \Rightarrow ma = -\lambda x$

$a = x''$. We get $x'' = -\mu x$, $\mu = -\frac{\lambda}{m}$

$\begin{cases} x'' + \lambda x = 0 \\ x(0)=1 \\ x'(0)=0\end{cases}$ $(\triangle)$

$x(t) = \sin(\sqrt{\lambda}t) \Rightarrow x''(t) = (x'(t))' = \left(\sqrt{\lambda}\cos(\sqrt{\lambda}t)\right)' = -(\sqrt{\lambda})^2\sin(\sqrt{\lambda}t) = -\lambda x(t)$.

$y(t) = \cos(\sqrt{\lambda}t) \Rightarrow y''(t) = -\lambda y(t)$.

- The equation is linear. (of the form $a(t)x'' + b(t)x' + c(t) \overset{\triangle}{=} 0$).
- The equation is of order 2. (The derivatives appear up to order 2)

Thm: If we have 2 linearly independent solution to $\triangle$
$\Rightarrow$ Any solution is obtained as a linear combination of the 2 given solutions.

def: 2 functions x,y are l.i. (linearly independent) if $\forall\alpha,\beta\in\mathbb{R}: \alpha x(t) + \beta y(t) = 0$ $\forall t\in D(x) = D(y) \Rightarrow \alpha=\beta=0$.

A linear combination of 2 functions x and y (with $D(x)=D(y)$) is an expression of the form $\alpha x(t) + \beta y(t)$, for some $\alpha,\beta\in\mathbb{R}$.

Any solution to $x''+\lambda x = 0$ can be written as $x = \alpha\cos(\sqrt{\lambda}t) + \beta\sin(\sqrt{\lambda}t)$, for some $\alpha,\beta\in\mathbb{R}$.
We want to satisfy $x(0)=1$, $x'(0)=0$.

$x(0) = \alpha\cdot 1 + \beta\cdot 0 = \boxed{\alpha = 1}$

$x'(t) = -\alpha\sqrt{\lambda}\sin(\sqrt{\lambda}t) + \beta\sqrt{\lambda}\cos(\sqrt{\lambda}t)$

$x'(0) = -\alpha\sqrt{\lambda}\cdot 0 + \beta\sqrt{\lambda}\cdot 1 = 0 \Rightarrow \beta\sqrt{\lambda} = 0 \Leftrightarrow \boxed{\beta = 0}$

Finally: $x(t) = \cos(\sqrt{\lambda}t)$