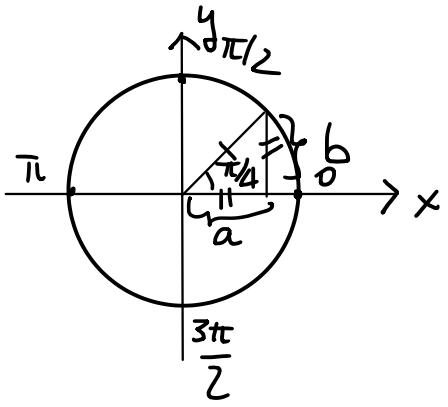
Lesson 30/10/2026

venerdì 30 ottobre 2026 08:52

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$$



$\vartheta = \frac{\pi}{4} \leftrightarrow \alpha = 45°$

$a = \cos\frac{\pi}{4}, \quad b = \sin\frac{\pi}{4}$

$a^2+b^2=1$

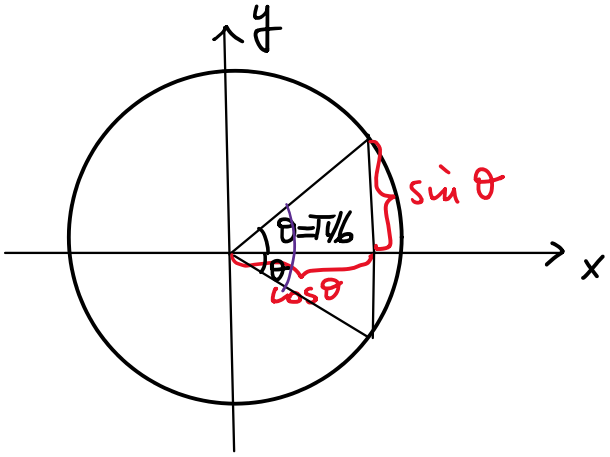
$$\Leftrightarrow \begin{cases} \cos\left(\frac{\pi}{4}\right)^2 + \sin\left(\frac{\pi}{4}\right)^2 = 1 \\ \cos\frac{\pi}{4} = \sin\frac{\pi}{4} \end{cases}$$

$2\cos^2\left(\frac{\pi}{4}\right) = 1 \Leftrightarrow \cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2} \Leftrightarrow \cos\left(\frac{\pi}{4}\right) = \pm\frac{1}{\sqrt{2}}$

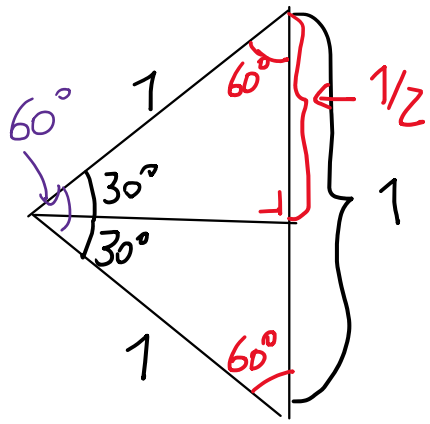
we need to choose $+$, so that $\boxed{\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}}$

$\Rightarrow \boxed{\sin\left(\frac{\pi}{4}\right) = 1/\sqrt{2}} \quad \left(= \frac{\sqrt{2}}{2}\right)$

$\vartheta = \frac{\pi}{6} \Leftrightarrow \alpha = 30°$



blue angle $= 60°$



the big triangle has 3 equal sides (and 3 equal angles).

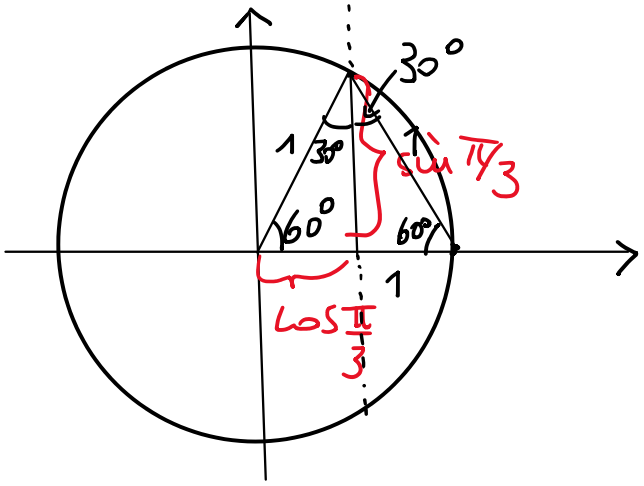
$$\Rightarrow \boxed{\sin\frac{\pi}{6} = \frac{1}{2}}$$

$$\cos\left(\frac{\pi}{6}\right)^2 + \sin\left(\frac{\pi}{6}\right)^2 = 1$$

$$\cos\left(\frac{\pi}{6}\right)^2 + \frac{1}{4} = 1$$

$$\cos\left(\frac{\pi}{6}\right)^2 = \frac{3}{4} \quad (\Leftrightarrow \boxed{\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}}$$

$$\theta = \frac{\pi}{3} \quad (\Leftrightarrow \alpha = 60^\circ$$



$$\boxed{\cos\frac{\pi}{3} = \frac{1}{2}}$$

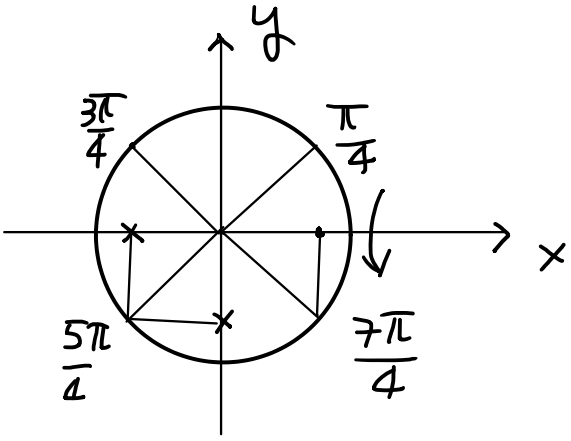
$$\sin\left(\frac{\pi}{3}\right)^2 + \cos\left(\frac{\pi}{3}\right)^2 = 1$$

$$\sin\left(\frac{\pi}{3}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\boxed{\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}}$$

Other "easy" angles:

Other "easy" angles:



$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

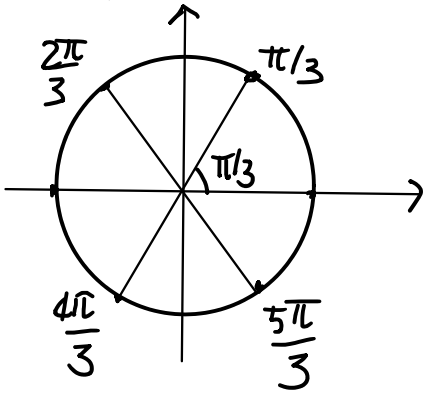
$$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

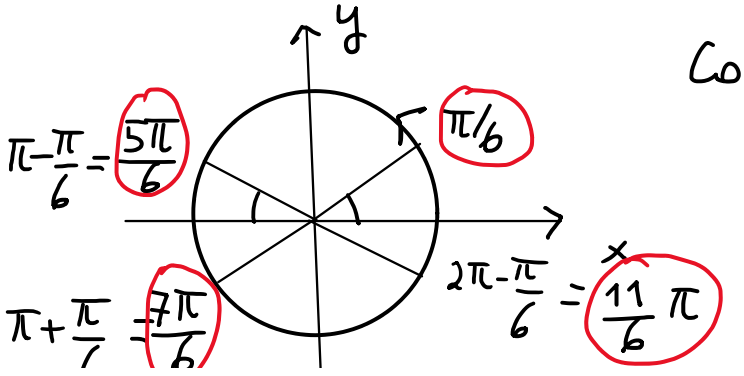
$$\cos\frac{5\pi}{4} = \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}$$

$$\cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}}, \quad \sin\frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$$

$\cos\frac{2\pi}{3}$, $\sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3}$, $\sin\frac{4\pi}{3}$, $\cos\frac{5\pi}{3}$, $\sin\frac{5\pi}{3}$



exercise.



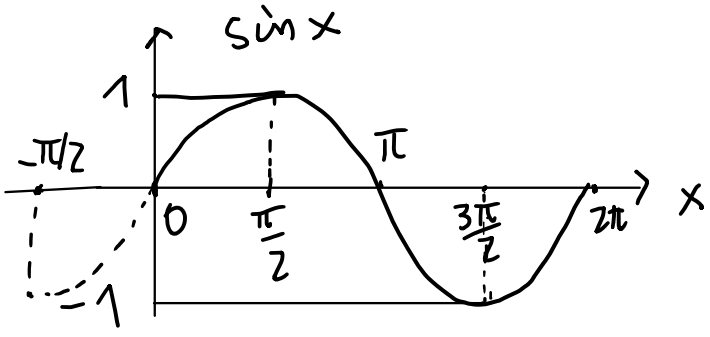
Compute the sin and the cos of the red angles.

$\pi + \frac{\pi}{6} = \frac{7\pi}{6}$ $\qquad 2\pi - \frac{\pi}{6} = \frac{11}{6}\pi$

Graphs of the sin and the cos

We observed that sin and cos are 2π-periodic. In particular, it is enough to draw the graph in the interval $[0, 2\pi]$ and then extend it by periodicity.

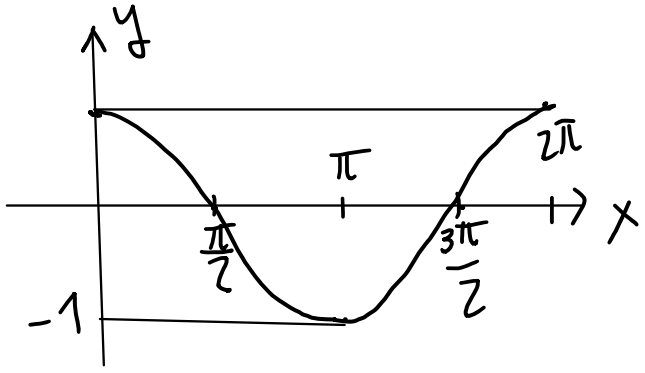
the sin:



Remark:

Note that the sin is increasing in $[0, \pi/2]$ and in $[\frac{3\pi}{2}, 2\pi]$ while it is decreasing between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

The cos:



the cos is decreasing in $[0, \pi]$ and increasing in $[\pi, 2\pi]$.

Inverse functions:

Question? Is the sin : $\mathbb{R} \to \mathbb{R}$ invertible?

Question? Is the $\sin : \mathbb{R} \to \mathbb{R}$ invertible?

Is the sin injective? The sin is 2π-periodic

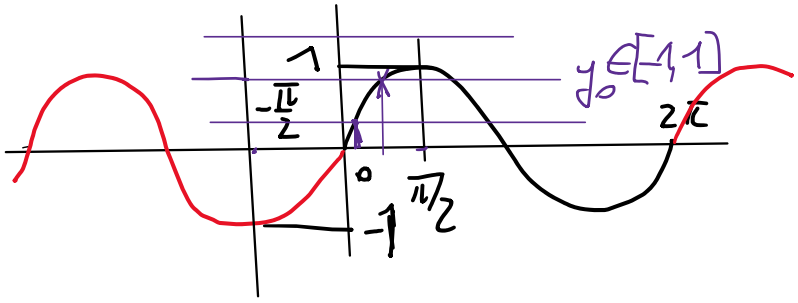
$\Rightarrow \sin(0) = 0 = \sin(2\pi) \Rightarrow$ sin is not injective on $\mathbb{R}$.

However, the sin is increasing in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$\Rightarrow \sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$ is injective $\Rightarrow$

$$\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]} : [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to \sin\left([-\tfrac{\pi}{2}, \tfrac{\pi}{2}]\right) = [-1, 1]$$

is bijective.



$\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$ is injective.

$\Leftrightarrow \forall y_0 \in \mathbb{R}$, $\exists$ at most 1 solution $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ to the equation $\sin x = y_0$

$|y_0| > 1 \Rightarrow \nexists$ solution

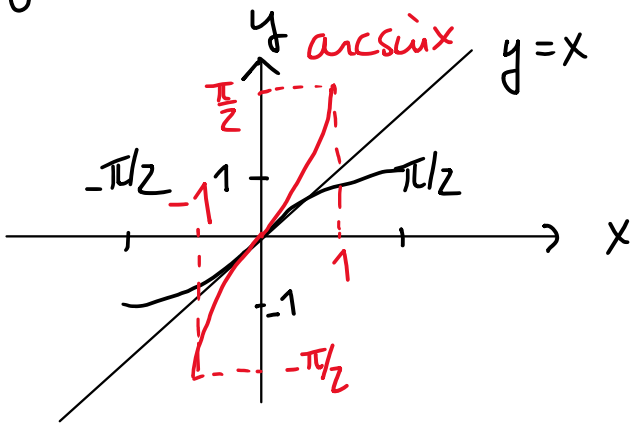
$|y_0| \leq 1 \Rightarrow \exists!$ solution.

In the end, we have proved that

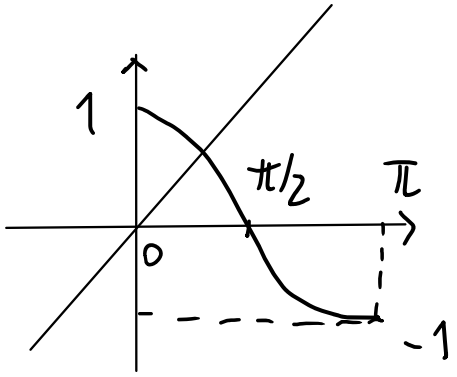
$$\sin|_{[-\frac{\pi}{2},\frac{\pi}{2}]} : [-\tfrac{\pi}{2},\tfrac{\pi}{2}] \to [-1,1]$$

is invertible. The inverse is known as the $\arcsin : [-1,1] \to [-\tfrac{\pi}{2},\tfrac{\pi}{2}]$.

The graph of the arcsin:



Similarly we can define the inverse of $\cos|_{[0,\pi]}$.



① $\cos|_{[0,\pi]}$ is decreasing

② $\Rightarrow \cos|_{[0,\pi]} : [0,\pi] \to \mathbb{R}$ is injective.

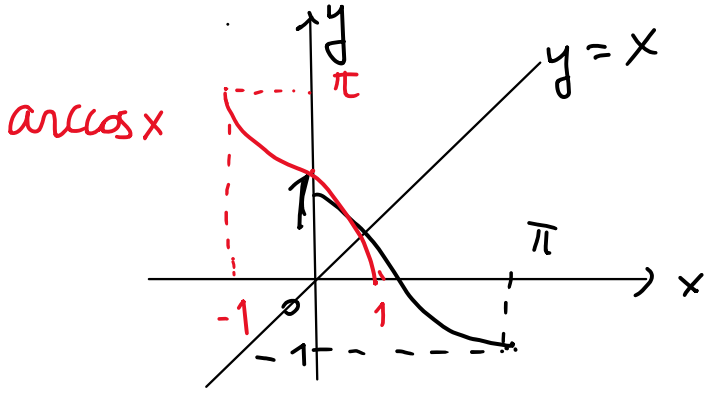
③ $\Rightarrow \cos|_{[0,\pi]} : [0,\pi] \to [-1,1]$ is surjective $\Rightarrow$ it is bijective.

④ $\Rightarrow \cos|_{[0,\pi]} : [0,\pi] \to [-1,1]$

is invertible.

The inverse is known as $\arccos : [-1,1] \to [0,\pi]$

Graph of the arccos:



Remarks: . sin, cos are bounded functions.

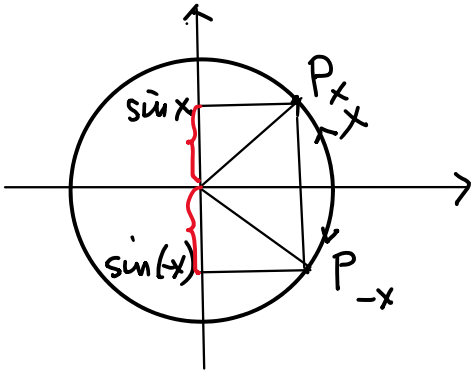
$\sin(\mathbb{R}) = \cos(\mathbb{R}) = [-1,1]$

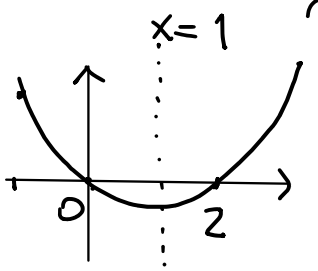
. sin is odd, cos is even

$$\sin(-x) = -\sin x \qquad \cos x = \cos(-x)$$

. $f(x) = 2^x$ is neither odd nor even.

. $f(x) = x^2 - 2x$ is neithe odd nor even.



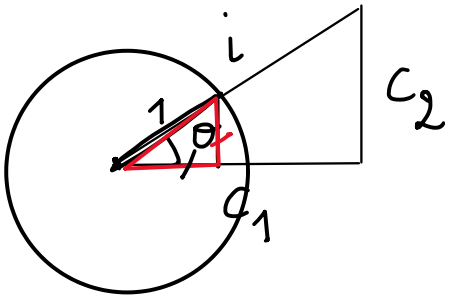


$f(0) = f(2) = 0$

$f > 0$ for $x < 0 \vee x > 2$

$f < 0$ for $0 < x < 2$

Theorems about rectangle triangles:

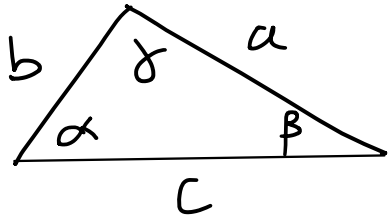


$$\begin{cases} c_1 = i\cos\theta \\ c_2 = i\sin\theta \end{cases}$$

$$\frac{c_1}{\cos\theta} = \frac{i}{1} \iff c_1 = i\cos\theta$$

$$\frac{c_2}{\sin\theta} = \frac{i}{1} \iff c_2 = i\sin\theta$$

Amore general result: (the sin theorem)

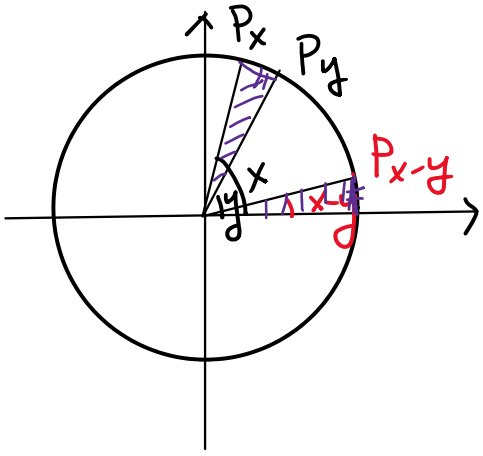


$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

The sin and the cos of a + or − of angles:

$$\sin(x\pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y$$

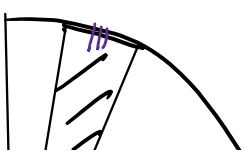


$$P_x = (\cos x, \sin x)$$

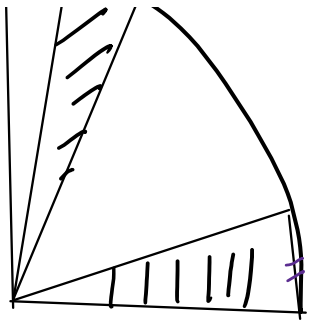
$$P_y = (\cos y, \sin y)$$

$$P_{x-y} = (\cos(x-y), \sin(x-y))$$

$$P_0 = (1,0).$$



$$|P_x - P_y|^2 = |P_{x-y} - P_0|^2$$



$$|e^{ix}-e^{iy}| = |e^{ix}-y| \quad |\cdot|$$

$$(\cos x-\cos y)^2+(\sin x-\sin y)^2=(\cos(x-y)-1)^2 + (\sin(x-y)-0)^2$$

$$(a+b)^2=a^2+2ab+b^2, \quad \forall a,b\in\mathbb{R}.$$

$$(\cos x)^2-2\cos x\cos y+(\cos y)^2+(\sin x)^2-2\sin x\sin y + (\sin y)^2 = \cos^2(x-y)-2\cos(x-y)+1+\sin(x-y)^2$$

$$\cos^2\alpha+\sin^2\alpha=1, \quad \forall \alpha\in\mathbb{R}$$

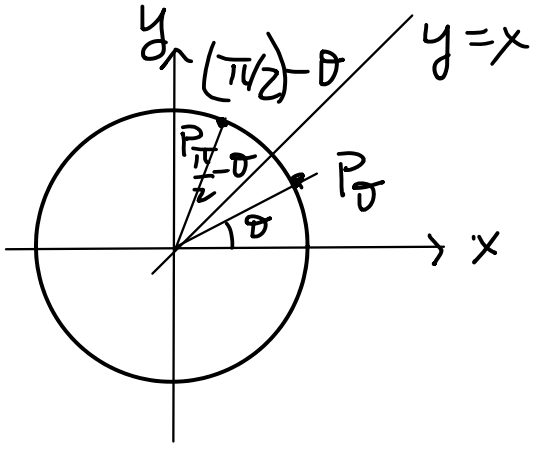
$$2-2\cos x\cos y-2\sin x\sin y = 1+1-2\cos(x-y)$$

$$\boxed{\cos(x-y)=\cos x\cos y+\sin x\sin y}$$

$$\cos(x+y)=\cos(x-(-y))=\cos x\cos(-y)+\sin x\sin(-y)$$

$$\boxed{\cos(x+y)=\cos x\cos y-\sin x\sin y.}$$

What about the sin!



$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

$$\sin(x+y) = \cos\left(\frac{\pi}{2} - (x+y)\right) = \cos\left(\left(\frac{\pi}{2} - x\right) - y\right)$$

$$= \cos\left(\frac{\pi}{2} - x\right)\cos y + \sin\left(\frac{\pi}{2} - x\right)\sin y$$

$$\boxed{\sin(x+y) = \sin x \cos y + \cos x \sin y}$$

$$\sin(x-y) = \sin(x + (-y)) = \sin x \cos(-y) + \cos x \sin(-y)$$

$$\boxed{\sin(x-y) = \sin x \cos y - \cos x \sin y}$$

- Sin and cosine of $2x$:

$$\sin(2x) = \sin(x+x) = \sin x \cos x + \cos x \sin x$$

$$\Rightarrow \quad \boxed{\sin 2x = 2 \sin x \cos x}$$

$$\cos(2x) = \cos(x+x) = \cos x \cdot \cos x - \sin x \cdot \sin x$$

$$\Rightarrow \quad \boxed{\cos 2x = \cos^2 x - \sin^2 x}$$

- sin and cos of $x/2$.

$$\cos 2x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad (\Rightarrow \quad \sin x = \pm\sqrt{\frac{1 - \cos(2x)}{2}}$$

$$2x = y \Rightarrow \quad x = y/2 \quad \Rightarrow \quad \boxed{\sin\left(\frac{y}{2}\right) = \pm\sqrt{\frac{1 - \cos y}{2}}}$$

+ if $y/2 \in$ I or II quadrant

+ if $y/2 \in$ I or II quadrant

− if $y/2 \in$ III or IV quadrant

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x) = 2\cos^2 x - 1$$

$$\cos^2 x = \frac{\cos(2x)+1}{2} \iff \cos x = \pm\sqrt{\frac{\cos(2x)+1}{2}}$$

$$x = \frac{y}{2} \iff \boxed{\cos\left(\frac{y}{2}\right) = \pm\sqrt{\frac{\cos y + 1}{2}}}$$

+ if $\frac{y}{2} \in$ I or IV quadrant

− if $\frac{y}{2} \in$ II or III quadrant

Application:

Compute the sin and the cos of some new angles: $\theta = \pi/12$.

$$\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$$

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$$

$$\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)$$

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$\vartheta = \frac{\pi}{8} = \frac{1}{2}\frac{\pi}{4}$ ...

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

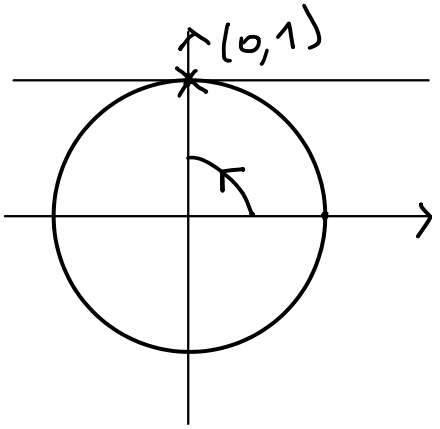
$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

Examples - exercises:

trigonometric equations:

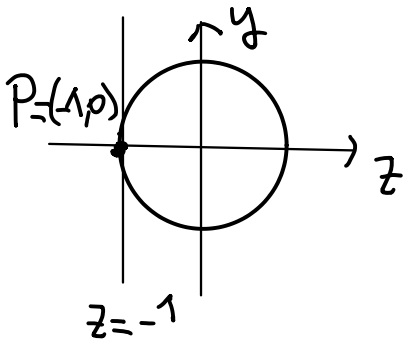


$\sin x = 1$

. which angle corresponds to the point $(0,1)$?

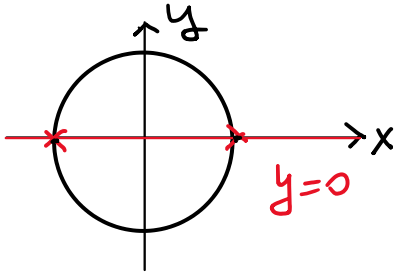
$x = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$

. $\cos x = -1$ $\quad x = \pi + 2k\pi$, $k \in \mathbb{Z}$.



- $\sin x = 0$ $\quad x = 2k\pi \ \vee \ x = -\pi + 2k\pi \quad k \in \mathbb{Z}$

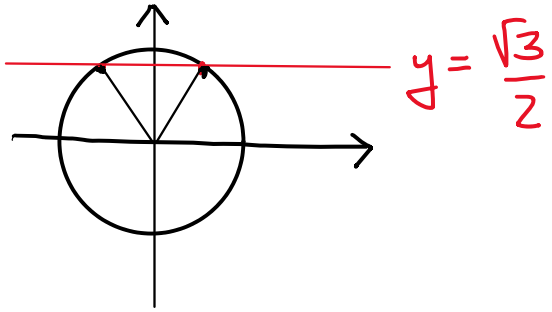
$$x = \pi(-1+2k)$$



$\iff$ $\boxed{x = k\pi, \ k \in \mathbb{Z}.}$

- $2\sin x = \sqrt{3} \iff \sin x = \frac{\sqrt{3}}{2} \iff x = \frac{\pi}{3} + 2k\pi \ \vee \ x = \frac{2\pi}{3} + 2k\pi$

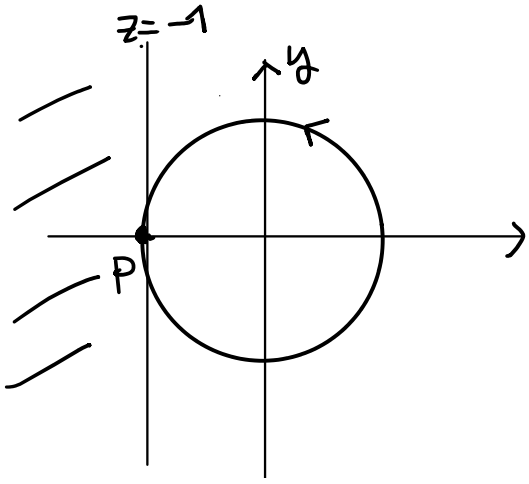
$k \in \mathbb{Z}$.



- $\sin x \geq 5$ impossible.

- $\cos x \leq -1$ we need to intersect the circle with the half-plane

$$\Pi = \{(x,y) \in \mathbb{R}^2 : x \leq -1\}$$



$$S^1 \cap \Pi = \{(-1,0)\} = \{P\}$$

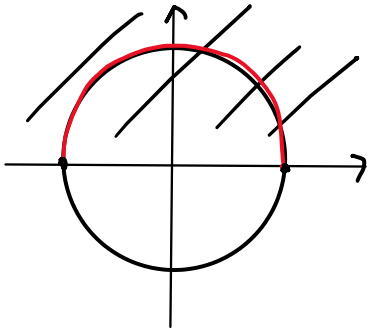
$P = (-1,0)$.

$x = \pi + 2k\pi \quad k \in \mathbb{Z}$

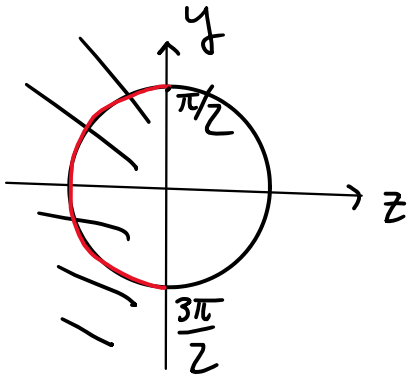
$x = \pi + 2k\pi \quad k \in \mathbb{Z}$

- $\sin x \geq 0 \qquad 0 + 2\pi k \leq x \leq \pi + 2\pi k \quad k \in \mathbb{Z}$

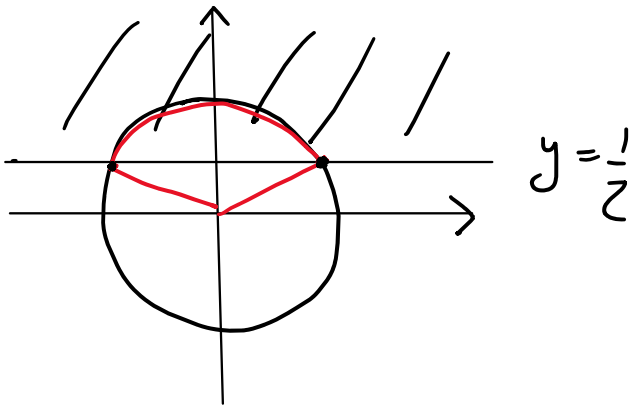
$2\pi k \leq x \leq \pi + 2k\pi \quad k \in \mathbb{Z}.$



- $\cos x < 0 \iff \frac{\pi}{2} + 2k\pi < x < \frac{3\pi}{2} + 2k\pi \quad k \in \mathbb{Z}.$



- $\sin x \geq \frac{1}{2} \iff \frac{\pi}{6} + 2k\pi \leq x \leq 2k\pi + \frac{5\pi}{6}$

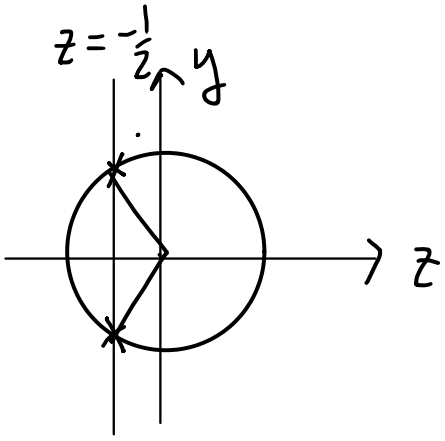


- $\sin x + \sin 2x = 0.$

$\sin x + 2 \sin x \cos x = 0$

$\sin x (1 + 2\cos x) = 0 \iff \sin x = 0 \ \vee \ 1 + 2\cos x = 0.$

$x = k\pi \ \vee \ \cos x = -\frac{1}{2}$



$$\boxed{x = k\pi \ \vee \ x = \frac{2\pi}{3} + 2k\pi \ \vee \ x = \frac{4\pi}{3} + 2k\pi}$$

- $\cos^2 x + \sin x = 1$

$1 - \sin^2 x + \sin x = 1$

$-\sin^2 x + \sin x = 0$

$\sin x (1 - \sin x) = 0 \iff \sin x = 0 \ \vee \ 1 - \sin x = 0$

$\iff x = k\pi \ \vee \ \sin x = 1$

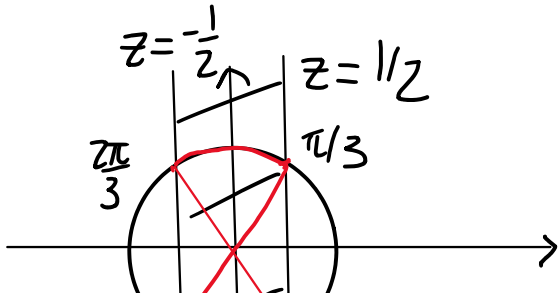
$\iff x = k\pi \vee x = \frac{\pi}{2} + 2k\pi.$

- $3\cos^2 x - \sin^2 x \leq 0$

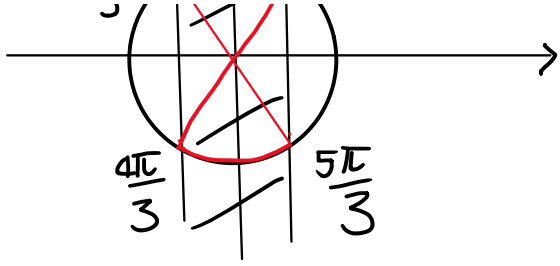
$3\cos^2 x - (1 - \cos^2 x) \leq 0$

$4\cos^2 x - 1 \leq 0$

$\cos^2 x \leq \frac{1}{4} \iff -\frac{1}{2} \leq \cos x \leq \frac{1}{2}$



$$\boxed{\frac{\pi}{3} + 2k\pi \leq x \leq \frac{2\pi}{3} + 2k\pi \qquad \frac{4\pi}{3} + 2k\pi \leq x \leq \frac{5\pi}{3} + 2k\pi}$$



$$\frac{4\pi}{3}+2k\pi \le x \le \frac{5\pi}{3}+2k\pi$$

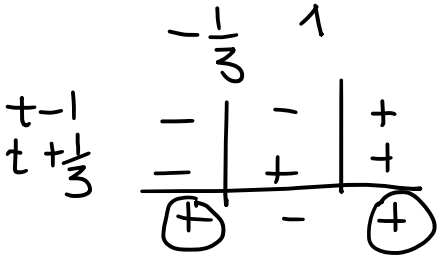
- $3\cos^2 x - 2\cos x + 1 > 0$

$t = \cos x$

$3t^2 - 2t - 1 > 0$

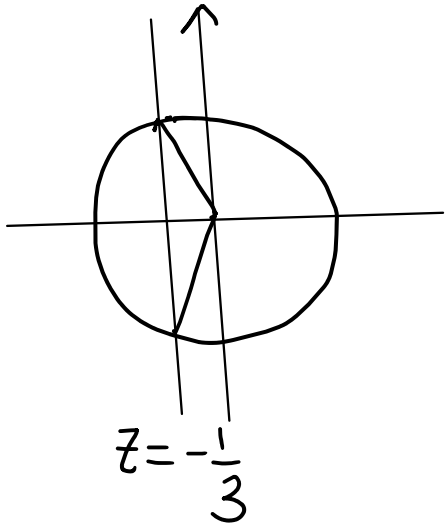
$$t_{1,2} = \frac{2 \pm \sqrt{4+4\cdot 3}}{3\cdot 2} = \frac{2 \pm \sqrt{4+12}}{6} = \frac{2 \pm 4}{6} = 1 \,/\, -\frac{1}{3}$$

$$3t^2 - 2t - 1 = 3(t-1)\left(t+\frac{1}{3}\right) > 0 \iff t > 1 \vee t < -\frac{1}{3}$$



$\cos x > 1 \vee \cos x < -\frac{1}{3}$

$\iff \cos x < -\frac{1}{3}$



$$\arccos\left(-\frac{1}{3}\right)+2k\pi < x < 2\pi - \arccos\left(-\frac{1}{3}\right)+2k\pi$$

$k \in \mathbb{Z}$.

- $2^{2x} - 5\cdot 2^x + 6 = 0$

- $2^{2x} - 5 \cdot 2^x + 6 = 0$

$4^x - 5 \cdot 2^x + 6 = 0$

$4^x = (2^x)^2$

$t = 2^x$, $t^2 - 5t + 6 = 0 \iff (t-2)(t-3) = 0$

$\iff t = 2 \vee t = 3$

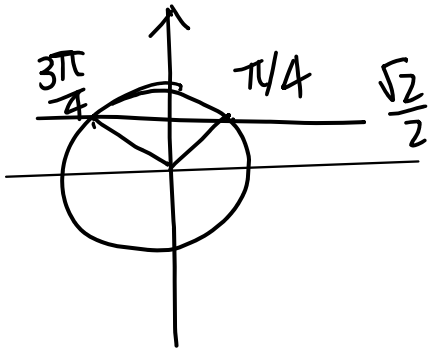
$\iff 2^x = 2 \vee 2^x = 3$

$\iff x = \log_2 2 = 1 \vee x = \log_2 3$

$\iff \boxed{x = 1 \vee x = \log_2 3}$

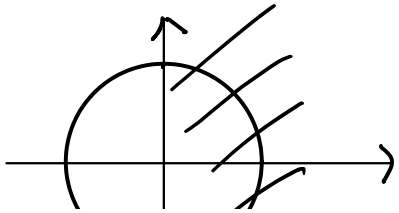
- $\dfrac{2\sin x + \sqrt{2}}{\cos x} \leq 0$

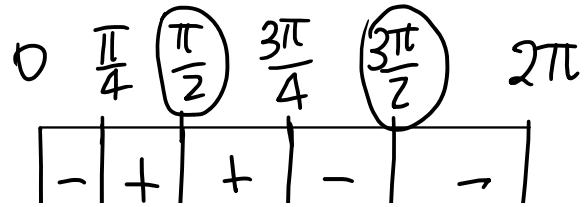
- $2\sin x + \sqrt{2} \geq 0 \iff \sin x \geq \dfrac{\sqrt{2}}{2} \iff$

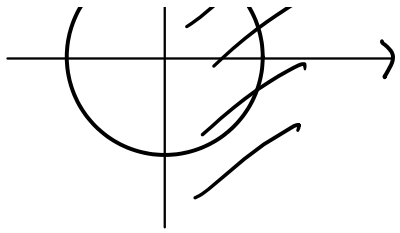


$\dfrac{\pi}{4} + 2\pi k \leq x \leq \dfrac{3\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$.

- $\cos x > 0 \iff 2\pi k < x < \dfrac{\pi}{2} + 2k\pi \vee \dfrac{3\pi}{2} + 2\pi k < x < 2\pi + 2k\pi$







N	−	+	+	−	−
D	+	+	−	−	+
	−	+	−	+	−

$2\pi k \le x \le \frac{\pi}{4} + 2k\pi \vee \frac{\pi}{2} + 2\pi k < x \le \frac{3\pi}{4} + 2k\pi \vee \frac{3\pi}{2} + 2\pi k < x \le 2\pi k$

- $\log\left(\frac{x^2-3}{1-3x}\right) > 0 \overset{es1}{\iff} \frac{x^2-3}{1-3x} > 1$

$\iff \frac{x^2-3}{1-3x} - 1 > 0 \iff \frac{x^2-3-(1-3x)}{1-3x} > 0$

$\frac{x^2+3x-4}{1-3x} > 0$

$\frac{(x+4)(x-1)}{1-3x} > 0$

$x+4=0 \iff x=-4 \; ; \; x-1=0 \iff x=1$

$\iff 1-3x=0 \iff x=1/3$

$-4 \quad \frac{1}{3} \quad 1$

−	+	+	+	
−	−	−	+	
+	+	−	−	
+	−	+	−	

$x \in (-\infty, -4) \cup \left(\frac{1}{3}, 1\right)$.

- $a^{-x} \cdot a^{x} \cdot 6 > 0 \qquad 2^x > 0$

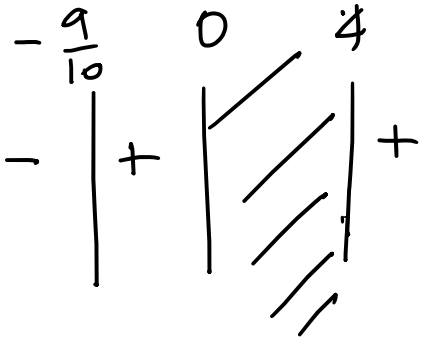
$2^{-x} + 2^x + 6 \geq 0 \quad , \quad 2^x > 0$

$\Rightarrow 2^{-x} = \frac{1}{2^x} > 0 \quad , \quad 6 > 0 \quad \Rightarrow \quad 2^{-x} + 2^x + 6 > 0$ $\boxed{\forall x \in \mathbb{R}}$

- $\sqrt{x^2 - 4x} \leq x + 3$

$$\begin{cases} x^2 - 4x \geq 0 \\ x^2 - 4x \leq (x+3)^2 \end{cases} \iff \begin{cases} x(x-4) \geq 0 \\ x^2 - 4x \leq x^2 + 9 + 6x \end{cases}$$

$$\iff \begin{cases} x \leq 0 \ \vee \ x \geq 4 \\ -9 \leq 10x \end{cases} \iff \begin{cases} x \leq 0 \ \vee \ x \geq 4 \\ x \geq -\frac{9}{10} \end{cases}$$



$\boxed{-\frac{9}{10} \leq x \leq 0 \quad \vee \quad x \geq 4}$