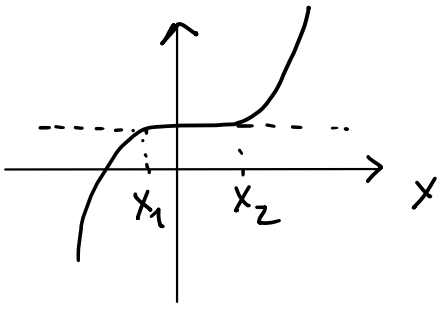
Lesson 29/10/2026

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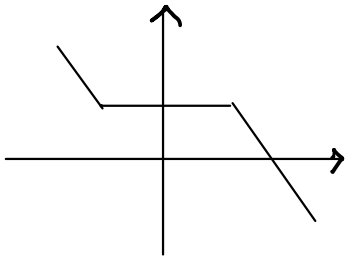
Qualitative properties of functions:

Def: A function $f: X \subset \mathbb{R} \to \mathbb{R}$ is said to be

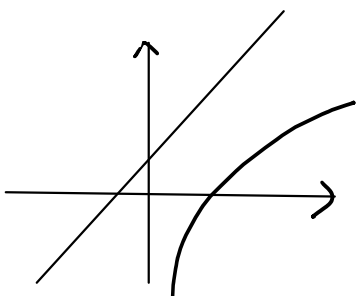
- monotone non decreasing if $\forall x_1, x_2 \in X$ with $x_1 \leq x_2$, we have $f(x_1) \leq f(x_2)$.
- monotone non increasing if $\forall x_1, x_2 \in X$, $x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$.
- monotone increasing (strictly increasing) if $\forall x_1, x_2 \in X$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- monotone decreasing (strictly decreasing) if $\forall x_1, x_2 \in X$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.



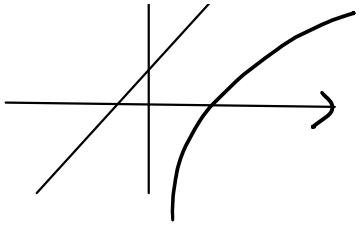
non-decreasing.

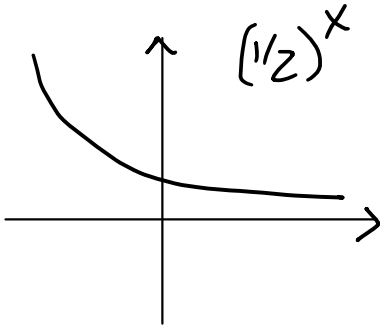


non-increasing.



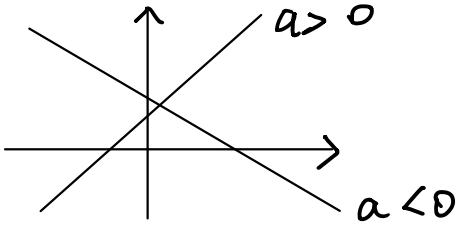
strictly increasing





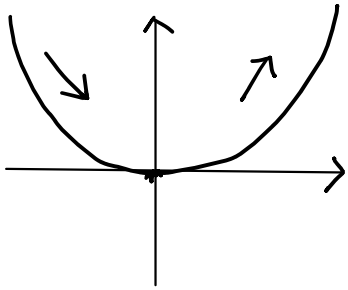
stictly decreasing.

examples: $y = ax+b$ is $\begin{cases} \text{stictly increasing if } a>0 \\ \text{stictly decreasing if } a<0 \end{cases}$



$y = a^x$, $\log_a x$ is stictly $\begin{cases} \text{increasing if } a>1 \\ \text{decreasing if } a<1 \end{cases}$

$f(x) = x^2$ is stictly $\begin{cases} \text{increasing in } (0,\infty) \\ \text{decreasing in } (-\infty,0). \end{cases}$



proposition: Let $I \subset \mathbb{R}$ be an interval.

If $f: I \to \mathbb{R}$ is stictly increasing (or stictly decreasing), then f is injective.

decreasing), then f is injective.

Proof: take $x_1, x_2 \in I$, $x_1 \neq x_2 \overset{?}{\Rightarrow} f(x_1) \neq f(x_2)$

Assume $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \Rightarrow f(x_1) \neq f(x_2)$

s.i.
(s.d.) $(f(x_1) > f(x_2))$ ▨

Remark: If $f: I \to \mathbb{R}$ is strictly increasing, then $f: I \to f(I)$ is invertible.

In fact, due to the above proposition

f increasing $\Rightarrow$ f is injective

$\Rightarrow f: I \to f(I)$ is also surjective

$\Rightarrow f: I \to f(I)$ is bijective $(\Leftrightarrow f: I \to f(I)$ is invertible.

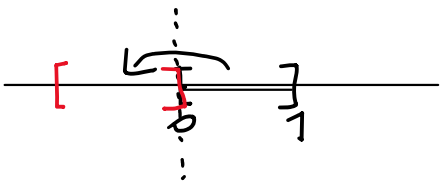
The same Remark holds true for strictly decreasing functions. ($f: I \to \mathbb{R}$ strictly decr $\Rightarrow f: I \to f(I)$ is invertible).

Symmetry:

Definition: let $X \subset \mathbb{R}$ be a subset.

we set $-X = \{-x : x \in X\}$

Example: $X = [0, 1] \Rightarrow -X = [-1, 0]$

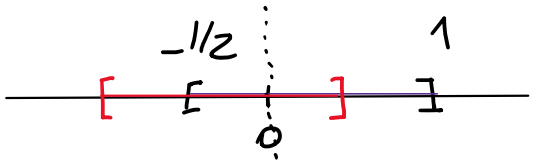


Definition: let $X \subset \mathbb{R}$ be a set. Then we

Definition: Let $X \subset \mathbb{R}$ be a subset. Then we say that X is symmetric with respect to the origin if $X = -X$.

In other words, a set $X \subset \mathbb{R}$ is symmetric w.r.to the origin, if it is invariant under reflection w.r.to the origin.

examples: $X = [-\frac{1}{2}, 1]$. Is X symmetric w.r.to 0?

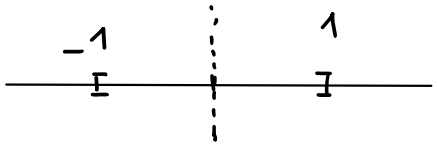


who is $-X$?

$$-X = \{-x : -\tfrac{1}{2} \le x \le 1\} = \{x : -1 \le x \le \tfrac{1}{2}\} = [-1, \tfrac{1}{2}]$$

$$-\tfrac{1}{2} \le x \le 1 \quad (\Leftrightarrow) \quad -1 \le -x \le \tfrac{1}{2}$$

- $X = [-1, 1]$



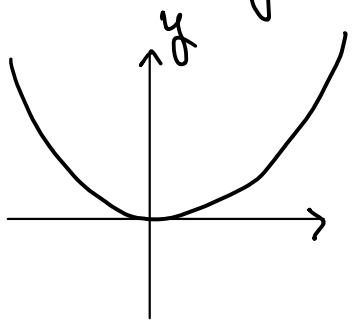
X is symmetric w.r. to 0.

Definition: Let $X \subset \mathbb{R}$ be a symmetric set (w.r.too). Let $f: X \to \mathbb{R}$ be a function.

(i) We say that f is even if $f(x) = f(-x)$, $\forall x \in X$.

(ii) We say that f is odd if $f(x) = -f(-x)$, $\forall x \in X$.

(ii) We say that f is odd if $f(x) = -f(-x)$, $\forall x \in X$.

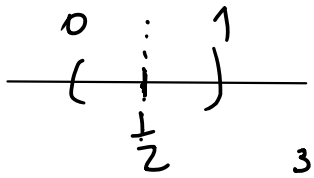


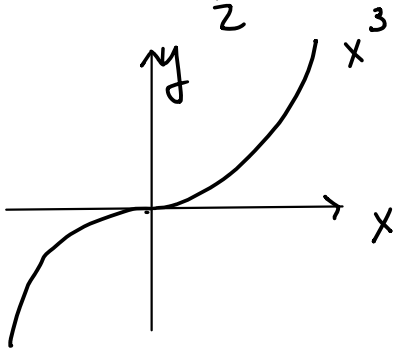
$f(x) = x^2$ is even. $f(-x) = f(x)$

$\iff (-x)^2 = x^2$. $\forall x \in \mathbb{R}$.

and $\mathbb{R}$ is symmetric w.r.to 0.

the set $(0,1)$ is symmetric w.r. to $\frac{1}{2}$





$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^3$ is odd.

$f(-x) = -f(x)$

$(-x)^3 = -(x^3)$

Remark: • a function f is even if its graph is symmetric with respect to the y-axis

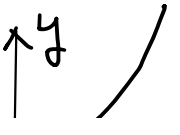
• a function f is odd if its graph is symmetric with respect to the origin.

• Given a number $n \in \mathbb{N} \Rightarrow n$ is either even or odd.

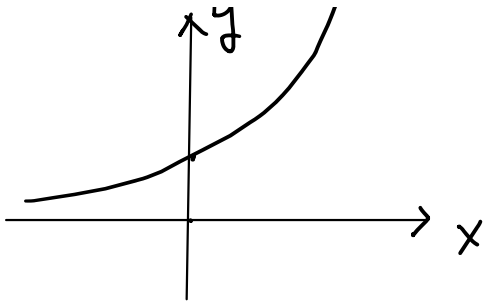
• Let $X \subset \mathbb{R}$ be a symmetric set (w.r. to 0).

Let $f : X \to \mathbb{R}$ be a function.

Question: Is it true that f is either even or odd?



$f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2^x$



$f:\mathbb{R}\to\mathbb{R}$, $f(x)=2^x$

$f(-x)=2^{-x}=\left(\frac{1}{2}\right)^x$

$f(-x)\neq f(x) \quad \forall x\neq 0$

$f(-x)\neq -f(x) \quad \forall x\in\mathbb{R}.$

$2^x=2^{-x} \iff 2^x=\frac{1}{2^x} \iff 2^{x+x}=1 \iff 2^{2x}=1$

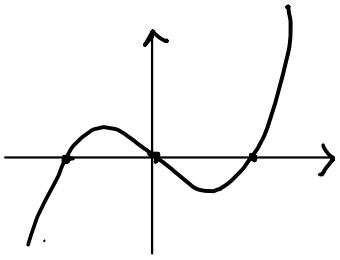
$\iff 2x=0 \iff x=0.$

Remark: Let $X\subset\mathbb{R}$ be symmetric w.r.t. 0. $[X\neq\{0\}]$
Let $f: X\to\mathbb{R}$ be an even function.
Then f is not injective.

proof: $\forall x\in X,\ f(x)=f(-x) \Rightarrow f$ not injective ∎

An odd function can be injective. For example x^3 is odd an injective.

question: $f: X\to\mathbb{R}$ is odd $\Rightarrow f$ is injective? No



$f(x)=x^3-x$ is odd

but not injective since

$f(1)=f(-1)=f(0)=0$

To sum up:

f even $\Rightarrow$ f not injective

f odd: we do not know if it is injective or not.

+ even $\Rightarrow$ + not injective

f odd: we do not know if it is injective or not.

[It may be injective, like x^3, or not, like $x^3 - x$]

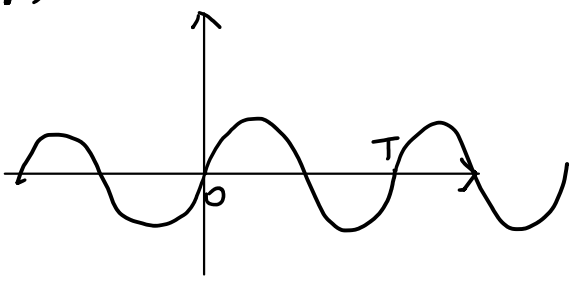
Periodic functions:

Def:

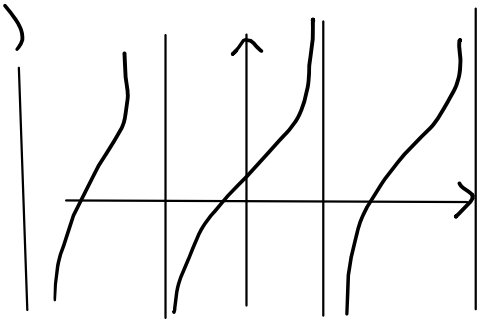
(i) Let $f: \mathbb{R} \to \mathbb{R}$ be a function. We say that f is periodic of period $T > 0$, if $f(x+T) = f(x)$, $\forall x \in \mathbb{R}$.

(ii) Let $x_0 \in \mathbb{R}$, $T > 0$. $f: \mathbb{R} \setminus \{x_0 + kT : k \in \mathbb{Z}\} \to \mathbb{R}$ is periodic if $f(x+T) = f(x)$, $\forall x \neq x_0 + Tk$, $k \in \mathbb{Z}$.

(i)



(ii)



Roughly: A function is periodic if it is enough to know the graph on a bounded interval and then extend it to the whole domain.

Proposition: Let $f: \mathbb{R} \to \mathbb{R}$ be periodic $\Rightarrow$ f is not injective.

proof: $f(x+T) = f(x)$ $\forall x \in \mathbb{R}$.

Bounded functions:

Bounded functions:

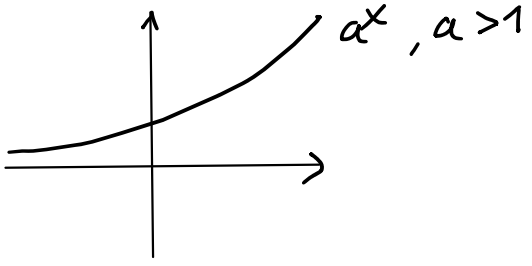
Definition: A function $f: X \subset \mathbb{R} \to \mathbb{R}$ is bounded if $f(X)$ is bounded.

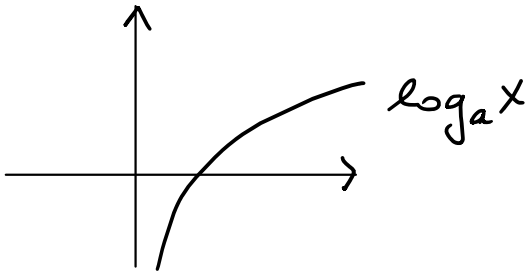
Definition: Let $f: X \subset \mathbb{R} \to \mathbb{R}$ be a function. We set

- $\inf_{x \in X} f(x) := \inf f(X)$
- $\sup_{x \in X} f(x) := \sup f(X)$

proposition: A function $f: X \subset \mathbb{R} \to \mathbb{R}$ is bounded $\iff \sup_{x \in X} f(x) \in \mathbb{R} \wedge \inf_{x \in X} f(x) \in \mathbb{R}$.

Examples: $f(x) = a^x$, $\log_a x$ are not bounded.





$$\inf_{x \in \mathbb{R}} a^x = 0 \leq \sup_{x \in \mathbb{R}} a^x = +\infty.$$

$a^x \geq 0 \quad \forall x \in \mathbb{R} \Rightarrow \inf_{x \in \mathbb{R}} a^x \geq 0.$

$$\inf_{x \in \mathbb{R}} \log_a x = -\infty < \sup_{x \in \mathbb{R}} \log_a x = +\infty.$$

Remark: · $\inf_{x\in X} f(x) \leq \sup_{x\in X} f(x)$

· $\inf_{x\in X} f(x) = \sup_{x\in X} f(x) \iff f$ is constant.

Definition · If $\exists x_0 \in X : \inf_{x\in X} f(x) = f(x_0)$, we say that x_0 is a minimizer and $f(x_0) = \min_{x\in X} f(x)$

· If $\exists x_0 \in X : \sup_{x\in X} f(x) = f(x_0)$ we say that x_0 is a maximizer and $f(x_0) = \max_{x\in X} f(x)$.

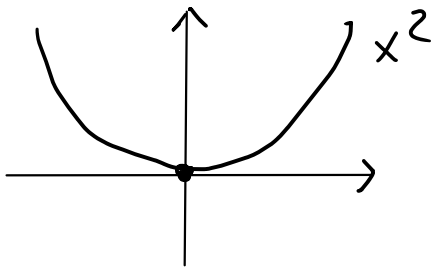
The min and the max are not always achieved.

Example: $f(x) = a^x$, $\inf_{x\in\mathbb{R}} f(x) = 0$. But f has no minimum. Because $f(x) > 0 \quad \forall x \in \mathbb{R}$.

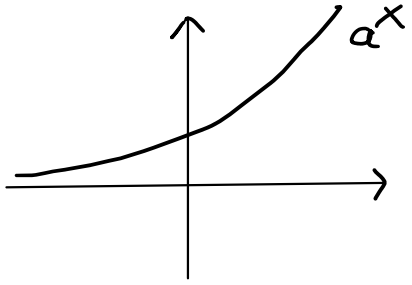
While the function $f(x) = x^2$ has a minimum, which is equal to 0 and is achieved at $x = 0$.

$f(0) = 0 \leq f(x), \quad \forall x \in \mathbb{R}$.

$f(0) = 0 \leq f(x), \quad \forall x \in \mathbb{R}.$



$\exists$ min



$\nexists$ min.

A similar argument shows that there are functions which have a maximum and functions which do not have a max.

Remark: $\inf_{x \in X} f(x) \in \mathbb{R} \not\Rightarrow \exists$ a min.

Definition: Let $f: X \to \mathbb{R}$ be a function.

We say that . f is bounded below if $\inf_{x \in X} f(x) \in \mathbb{R}$

. f is bounded above if

$\sup_{x \in X} f(x) \in \mathbb{R}$.

Example: . a^x is bounded below, but not bounded above.

. $-a^x$ is bounded above, but not bounded below.

Remark: $f: X \to \mathbb{R}$ is bounded $\Leftrightarrow$ f is both bounded below and bounded above.

bounded below and bounded above.

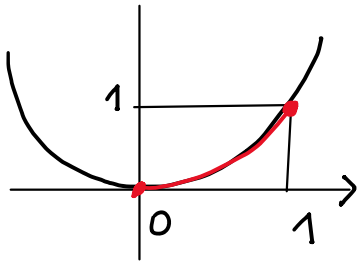
$\Leftrightarrow \inf_{x\in X} f(x) \in \mathbb{R} \wedge \sup_{x\in X} f(x) \in \mathbb{R}$.

Definition: $f: X \to \mathbb{R}$ is unbounded if it is not bounded.

Example: a^x, $\log_a x$, $3x+2$, x^2, x^3 are unbounded.

Remark: $f(x) = x^2$, $g := f|_{[0,1]} : [0,1] \to [0,1]$

$\Rightarrow g$ is bounded.

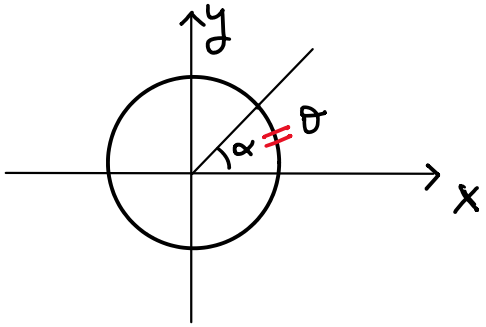


$\inf_{x\in[0,1]} g(x) = 0 = g(0) = \min_{[0,1]} g$

$\sup_{x\in[0,1]} g(x) = 1 = g(1) = \max_{[0,1]} g$

Trigonometry:

let $S^1 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$



$$\frac{\theta}{\alpha} = \frac{\text{length of } S^1}{360^\circ} = \frac{2\pi r}{360^\circ}$$

$$\pi = \frac{\text{length of circle}}{\text{diameter}} = \frac{\text{length}}{2r}$$

$$\pi = \frac{\text{length of circle}}{\text{diameter}} = \frac{\text{length}}{2r}$$

$$\Leftrightarrow \text{length} = 2\pi r.$$

$$r = 1 \Rightarrow \frac{\theta}{\alpha} = \frac{2\pi}{360^\circ} \Leftrightarrow \boxed{\theta = \frac{2\pi}{360^\circ} \cdot \alpha.}$$

the function $\alpha \in \mathbb{R} \mapsto \theta = \frac{2\pi}{360^\circ} \cdot \alpha$ is invertible.

For this reason, we have 2 equivalent ways of measuring angles.

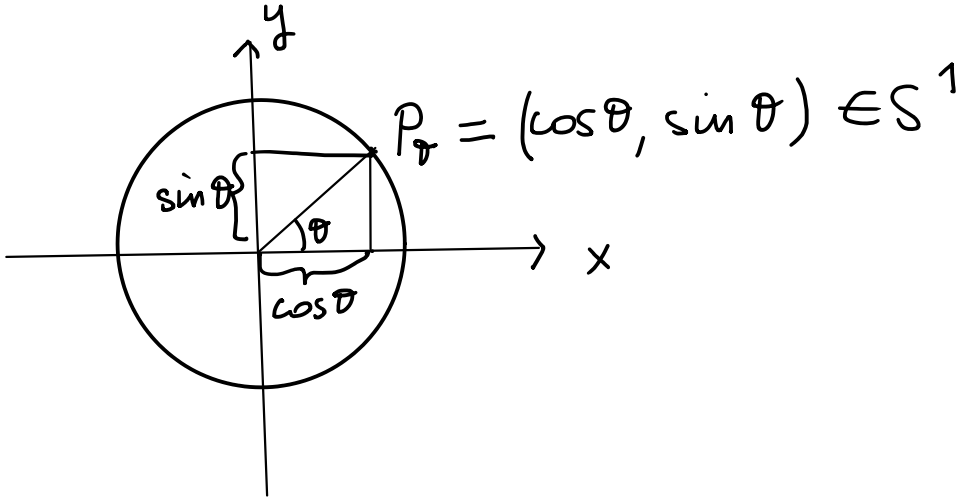
Definition: We call the length θ "radiants".

We can say either "the angle α measures 45°" or "the angle α measures $\frac{2\pi}{360^\circ} \cdot 45^\circ = \frac{\pi}{4}$ radiants".

degrees radiants

α	θ
0	$\frac{2\pi}{360^\circ} \cdot 0 = 0$
90°	$\frac{2\pi}{360^\circ} \cdot 90^\circ = \frac{\pi}{2}$
180°	$\frac{2\pi}{360^\circ} \cdot 180^\circ = \pi$
270°	$\frac{2\pi}{360^\circ} \cdot 270^\circ = \frac{3\pi}{2}$
360°	2π
45°	$\frac{2\pi}{360^\circ} \cdot 45^\circ = \pi/4$

$45°$	$\frac{2\pi}{360°} \cdot 45° = \pi/4$
$30°$	$\pi/6$
$60°$	$\pi/3$



$\cos: \mathbb{R} \to \mathbb{R}$, $\sin: \mathbb{R} \to \mathbb{R}$

$\theta \mapsto \cos\theta$ $\quad\quad$ $\theta \mapsto \sin\theta$.

Remark: $\forall\, \theta \in \mathbb{R}$, $-1 \le \cos\theta \le 1$, $-1 \le \sin\theta \le 1$.

In particular, the functions cos and sin are bounded.

$\inf_{x\in\mathbb{R}} \sin x = -1 < \sup_{x\in\mathbb{R}} \sin x = 1$

More precisely, $\sin\frac{3\pi}{2} = -1$, $\sin\frac{\pi}{2} = 1$.

The inf is a min, the sup is a max.

Similarly, $\cos 0 = 1 = \sup_{x\in\mathbb{R}} \cos x$

$\cos\pi = -1 = \min \cos x$

$\cos \pi = -1 = \min_{x \in \mathbb{R}} \cos x$.

Remark: sin, cos are periodic of 2π.

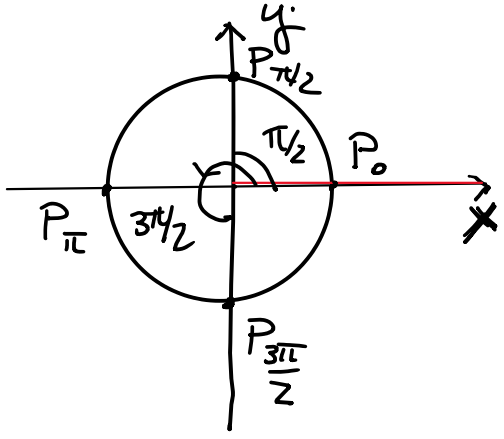
$\sin(x+2\pi) = \sin(x) \quad \forall x \in \mathbb{R}$

$\cos(x+2\pi) = \cos(x) \quad \forall x \in \mathbb{R}$.

Proposition: $\cos^2\theta + \sin^2\theta = 1 \quad , \forall \theta \in \mathbb{R}$

because the point $P_\theta = (\cos\theta, \sin\theta) \in S^1$

Cosine and sine of some well-known angles:



$\theta = 0$

$P_0 = (1, 0)$

$\cos 0 \quad \sin 0$

$\cos 0 = 1, \sin 0 = 0$

- $\theta = \pi/2$, $P_{\pi/2} = (0,1) \Rightarrow \cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1$.
- $\theta = \pi$, $\cos\pi = -1$; $\sin\pi = 0$.
- $\theta = \frac{3\pi}{2}$, $\cos\frac{3\pi}{2} = 0$; $\sin\frac{3\pi}{2} = -1$

We are looking for the intersection between the circle $\{x^2 + y^2 = 1\}$ and the half line $\{(x,y) : y < 0\}$.

$$\begin{cases} x^2+y^2=1 \\ x=0 \\ y<0 \end{cases} \iff \begin{cases} x=0 \\ y^2=1 \\ y<0 \end{cases} \iff \begin{cases} x=0 \\ y=\pm 1 \\ y<0 \end{cases}$$

$$\iff \begin{cases} x=0 \\ y=-1 \end{cases} \iff \begin{cases} \cos\frac{3\pi}{2}=0 \\ \sin\frac{3\pi}{2}=-1 \end{cases} \qquad \begin{cases} x=\cos\frac{3\pi}{2} \\ y=\sin\frac{3\pi}{2} \end{cases}$$

$\theta = 2\pi$. $\cos 2\pi = 1$, $\sin 2\pi = 0$.

- Friday 30th, october, lesson-Exercises from 9:00 am. to 12:30 p.m.
- Monday November 2nd: Normal schedule.

  9:00 am - 11:00 am Maths

  11:00 am - 1:00 pm Physical geography.
- Last lecture before Christmas: Friday 18th December
- 1st lecture after Christmas: Monday January 11th.